Equilibria and global dynamics of a problem with bifurcation from infinity

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Abstract: We consider a parabolic equation $u_t - \Delta u + u = 0$ with nonlinear boundary conditions $\frac{\partial u}{\partial n} = \lambda u + g(\lambda, x, u)$, where $\dfrac{g(\lambda, x, s)}{s} \to 0$ as $|s| \to \infty$. In [6] the authors proved the existence of unbounded branches of equilibria for λ close to an Steklov eigenvalue of odd multiplicity. In this work, we characterize the stability of such equilibria and analyze several features of the bifurcating branches. We also investigate several question related to the global dynamical properties of the system for different values of the parameter, including the behavior of the attractor of the system when the parameter crosses the first Steklov eigenvalue and the existence of extremal equilibria. We include an appendix where we prove a uniform antimaximum principle and several results related to the spectral behavior when the potential at the boundary is perturbed.

Keys words : stability, uniqueness, Steklov eigenvalues, bifurcation from infinity, sublinear boundary conditions, attractors, extremal equilibria, antimaximum principle.

1 Introduction

In this work we consider nonlinear parabolic equation with nonlinear boundary conditions

$$\begin{cases} u_t - \Delta u + u = 0, & \text{in } \Omega, \quad t > 0 \\ \frac{\partial u}{\partial n} = \lambda u + g(\lambda, x, u), & \text{on } \partial\Omega, \ t > 0 \\ u(0,x) = u_0(x), & \text{in } \Omega, \end{cases} \tag{1.1}$$

in a bounded and sufficiently smooth domain $\Omega \subset \mathbb{R}^N$ with $N \geq 2$ and analyze the behavior and stability properties of the equilibrium solutions as well as some features of the global dynamics. The equilibria are the solutions of the following elliptic problem with nonlinear boundary conditions

$$
\begin{cases} -\Delta u + u &= 0, \qquad\qquad \text{in } \Omega \\ \frac{\partial u}{\partial n} &= \lambda u + g(\lambda, x, u), \qquad \text{on } \partial\Omega. \end{cases} \tag{1.2}
$$

The main hypothesis on the nonlinearity g is the sublinearity at infinity with respect to the variable u. We will assume, roughly speaking, that

$$
|g(\lambda, x, u)| = o(|u|) \text{ as } |u| \to \infty.
$$

Hence, the boundary condition is asymptotically linear at infinity, since the dominant term for large values of $|u|$ is the linear term λu.

This problem was considered in [6] (see Theorem 3.3 of that paper). Let us denote by σ a Steklov eigenvalue, that is, a solution of

$$
\begin{cases} -\Delta \Phi + \Phi &= 0, \qquad \text{in } \Omega \\ \frac{\partial \Phi}{\partial n} &= \sigma\Phi, \qquad \text{on } \partial\Omega. \end{cases} \tag{1.3}
$$

We showed in [6] that some solutions of (1.2) become unbounded in Ω, when $\lambda \to \sigma$ and σ is of odd multiplicity. This is interpreted as a parametric resonance at the boundary. Even more, at the first Steklov eigenvalue, $\sigma = \sigma_1$, which is simple, the *branch* of unbounded solutions of (1.2) has two *subbranches* of, respectively, positive and negative equilibria, which moreover become unbounded everywhere in Ω; see Theorem 3.4 in [6]. In fact, for some continuum of solutions of (1.2), that we denote by u_λ, we have that

$$
\frac{u_\lambda(x)}{\|u_\lambda\|_{L^\infty(\partial\Omega)}} \to \pm\Phi_1(x), \quad \text{in} \quad C^\beta(\overline{\Omega}\,) \quad \text{as} \quad \lambda \to \sigma_1 \tag{1.4}
$$

for some $0 < \beta < 1$ and where $\Phi_1(x) > 0$ denotes the first positive Steklov eigenfunction, normalized in $L^\infty(\partial\Omega)$; see Corollary 3.2 in [6]. The choice of the sign depends on whether the *subbranch* is made of positive or negative equilibria. Note also that Φ_1 is strictly positive in $\overline{\Omega}$. In particular, from this we have

$$
\inf_{x\in\overline{\Omega}} |u_\lambda(x)| \to \infty, \quad \text{as} \quad \lambda \to \sigma_1. \tag{1.5}
$$

On the other hand, for λ far away from the Steklov eigenvalues, the set of solutions of (1.2) is nonempty and bounded in $\overline{\Omega}$, uniformly in λ. Also, as $\lambda \to \sigma_1$ equilibrium solutions that do not satisfy (1.4), remain bounded in $\overline{\Omega}$.

In the terminology of Bifurcation Theory, we say that, as $\lambda \to \sigma_1$, the unbounded branches of solutions of (1.2), u_λ, *bifurcate from infinity*, and that there exists a bifurcation from infinity at σ_1; see [14].

Also, some conditions were given in [6], which take into account the behavior of the nonlinearity g for $|u|$ large, which allows us to distinguish whether the unbounded branch of solutions of (1.2) (either positive or negative) is *subcritical* (that is only defined for parameter values λ *to the left* of σ_1,) or *supercritical* (that is *to the right* of σ_1). See Theorem 4.3 in [6] and Section 2 below for more details.

When these conditions imply that the whole *unbounded branch* of solutions of (1.2) is on one side of σ_1 only, one gets that the resonant problem, that is (1.2) for $\lambda = \sigma_1$, also has a solution.

This situation is guaranteed by some Landesman–Lazer type conditions, see Theorem 5.1 in [6] and [12].

As for the parabolic problem (1.1), it was shown in [6, Section 7] that when $\lambda < \sigma_1$, (1.1) is a dissipative system and has a global attractor, $\mathcal{A}_\lambda$. It was also shown in Proposition 7.1 in [6] that when the unbounded branch of positive equilibria is subcritical, there exists, for each fixed λ close enough to σ_1, the largest of such equilibria, which is asymptotically stable from above. Here stability is understood in the Lyapunov sense with respect to the parabolic problem (1.1). With an extra restriction, it was also shown that there exists the smallest of such large equilibria, which is asymptotically stable from below. An analogous result is obtained for the negative unbounded branch. Obviously, when there is a unique equilibria for fixed λ in such unbounded branch, then it is globally asymptotically stable with respect to initial data in (1.1), which are large everywhere in Ω.

In this paper, we proceed further in analyzing the structure and properties of unbounded branches of solutions of the elliptic problem (1.2) and on the global dynamics of the parabolic problem (1.1), when λ crosses σ_1.

First, we give conditions, which involve a more detailed knowledge of the behavior of the nonlinear term as $|u| \to \infty$, which imply that the unbounded branch of positive equilibria is subcritical, unique and stable, see Theorem 3.4. In an almost exact complementary situation, we also show that the unbounded branch of positive equilibria is supercritical, unique and unstable, see Theorem 3.5.

We also give conditions on the nonlinear term, which guarantee that the unbounded branch of positive solutions of (1.2) is monotonic in λ. This applies in particular, when the nonlinear term is of the form $\lambda g_0(x,u)$, with g_0 sublinear at infinity, that is, $|g_0(x,u)| = o(|u|)$ as $|u| \to \infty$. To get this monotonicity results, we need a uniform antimaximum principle for the linear Steklov problem, which is written in the appendix at the end of the article. We believe that the result in the Appendix is interesting by itself. We show that if we consider the linear nonhomogeneus problem

$$\left\{ \begin{array}{rcll} -\Delta u + u & = & 0, & \text{in } \Omega \\ \frac{\partial u}{\partial n} + b(x)u & = & \lambda u + g(x), & \text{on } \partial\Omega \end{array} \right. \tag{1.6}$$

then there exists a small $\delta > 0$ such that the antimaximum principle holds in $\mu_1(b) < \lambda < \mu_1(b) + \delta$, where $\mu_1(b)$ is the first Steklov eigenvalue associated to (1.6) (that is, the smallest λ for which there exists a solution of (1.6) with $g \equiv 0$). The parameter δ can be chosen uniformly for all potentials $b(x)$ lying in a small neighborhood of a given fixed potential $b_0(x)$ and also uniformly in $g(x)$ in certain sense, see Theorem A.3 in the Appendix for more details.

Let us mention that all these results, which are described in the introduction for positive solutions, have analogous statements for the negative branch of solutions.

For the parabolic problem (1.1) we give here a more complete picture of the global dynamics. In fact, as mentioned before, when $\lambda < \sigma_1$, (1.1) is a dissipative system and has a global attractor, $\mathcal{A}_\lambda$, see [6, Section 7]. Moreover there exist extremal equilibria in the sense of [15]. That is, there exists a pair of ordered equilibria which enclose any other equilibria as well as all the asymptotic dynamics of (1.1). These extremal equilibria are the *caps* of the attractor. See Lemma 5.1.

On the other hand, when $\lambda > \sigma_1$ then (1.1) is no longer dissipative and in fact there are initial conditions for which the solution (1.1) grows without bounds (blows-up in infinite time). To see this we just need to take an initial condition $u_0(x) \equiv M$ a very large constant. Hence, the character of the global dynamics changes drastically when λ crosses this value of the parameter and we want to understand how this affects the behavior of the attractors.

To analyze in detail the behavior of the attractors when the parameter crosses σ_1, we consider a nonlinear term of the form $\lambda g(x,u)$ with g sublinear at infinity, and assume the unbounded branches of positive and negative equilibria are supercritical. Then we show that any solution lying in the unbounded branches of positive and negative equilibria (which, from the results in Sections 3 and 4 are unique, monotonically decreasing in λ and unstable) have only one unstable eigenvalue. Even though, the systems is not dissipative for $\lambda > \sigma_1$, we prove the existence of a local attractor $\mathcal{A}_\lambda$, with $\lambda > \sigma_1$, with a very large basin of attraction. From here we get the existence of an attractor for the resonant case $\lambda = \sigma_1$. Note that this result can be interpreted as a Landesman–Lazer type result for attractors. Even more the attractors $\mathcal{A}_\lambda$ for $\lambda \leq \sigma_1$ and the local attractors $\mathcal{A}_\lambda$ for $\lambda > \sigma_1$ behave in an uppersemicontinuous way in λ. Furthermore the local attractors $\mathcal{A}_\lambda$ for $\lambda > \sigma_1$ have also extremal solutions.

The paper is organized as follows. In Section 2 we make precise the hypotheses on the nonlinearity and collect some notations and known results. We also give a more precise description of some of the results in the paper. Section 3 contains our stability results for the solutions of (1.2). In Section 4 we state sufficient conditions for monotonicity of the solutions of (1.2) with respect to the parameter. Section 5 is devoted to the global dynamics of the solutions of (1.1) for λ close to σ_1. The Appendix contains a proof of the uniform Antimaximum Principle and also several technical results on the behavior of the Steklov eigenvalues under variations of the potential at the boundary, which are needed in the paper and to show the uniform Antimaximum Principle.

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2 Preliminaries and description of the results

In this section we review the setting and results from [6], which we take as a starting point for our analysis. We also describe in a more technical and detailed way our results.

With respect to the nonlinearity g in (1.1) and (1.2), we assume the hypotheses

(H1) $g : \mathbb{R} \times \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathèodory function (i.e. $g = g(\lambda, x, s)$ is measurable in $x \in \Omega$, and continuous with respect to $(\lambda, s) \in \mathbb{R} \times \mathbb{R}$). Moreover, there exist $h \in L^r(\partial\Omega)$ with $r > N-1$ and continuous functions $\Lambda : \mathbb{R} \to \mathbb{R}^+$, $U : \mathbb{R} \to \mathbb{R}^+$, satisfying

$$|g(\lambda, x, s)| \leq \Lambda(\lambda) h(x) U(s), \qquad \forall (\lambda, x, s) \in \mathbb{R} \times \partial\Omega \times \mathbb{R}. \tag{2.1}$$

(H2) The function $U(s)$ satisfies

$$\lim_{|s|\to\infty} \frac{U(s)}{s} = 0, \tag{2.2}$$

that is, the function $g(\lambda, x, s)$ is sublinear at infinity in the variable s.

(H3) The nonlinearity $g(\lambda, x, s)$ is differentiable in s and

$$\frac{\partial g}{\partial s}(\lambda, \cdot, \cdot) \in C(\partial\Omega \times \mathbb{R}). \tag{2.3}$$

Elliptic regularity results and bootstrap arguments imply that solving (1.2) in, say $H^1(\Omega)$, is equivalent to solving the problem in a more regular space like Hölder spaces, see [6]. Hence, we may consider the solution pair (λ, u) of (1.2) in $\mathbb{R} \times C(\bar{\Omega})$. Since g is sublinear at infinity, the linear part of the boundary condition of (1.2) is the dominant term for u large enough. Hence, it is expected that large solutions of (1.2) can only exist, due to parametric resonance at the boundary, that is, when λ is near a Steklov eigenvalue, see (1.3). As mentioned in the Introduction this was actually proved in [6, Proposition 3.1, Theorem 3.3], at an eigenvalue of odd multiplicity. In particular this holds at σ_1, which is the case we consider in this paper. These results were obtained by showing that bifurcation from infinity occurs at such eigenvalues, see [14]. Furthermore we have (1.4) and (1.5).

To ellucidate whether or not the unbounded branch of solutions of (1.2) is subcritical or supercritical, the following quantities, which measure the asymptotic behavior of the nonlinear term at infintiy, were used, see [6, Theorems 4.3 and 4.5]:

$$\underline{\mathbf{G}}_+ := \int_{\partial\Omega} \liminf_{(\lambda,s)\to(\sigma_1,+\infty)} \frac{sg(\lambda,\cdot,s)}{|s|^{1+\alpha}} \Phi_1^{1+\alpha}$$

and

$$\overline{\mathbf{G}}_+ := \int_{\partial\Omega} \limsup_{(\lambda,s)\to(\sigma_1,+\infty)} \frac{sg(\lambda,\cdot,s)}{|s|^{1+\alpha}} \Phi_1^{1+\alpha},$$

for some $\alpha < 1$. It is shown in [6] that, if $\underline{\mathbf{G}}_+ > 0$, the positive unbounded branch of equilibria is subcritical, while it is supercritical if $\overline{\mathbf{G}}_+ < 0$.

To determine the stability of the solutions u_λ of (1.2) bifurcating from infinity at the first Steklov eigenvalue, σ_1, one must determine the sign of the first eigenvalue, Λ_1, of the linearized problem

$$\begin{cases} -\Delta\xi + \xi &= \Lambda\xi, & \text{in } \Omega \\ \frac{\partial\xi}{\partial n} &= \lambda\xi + g_u(\lambda, x, u_\lambda)\xi, & \text{on } \partial\Omega \end{cases}$$

where $g_u = \frac{\partial g}{\partial u}$, as $\lambda \to \sigma_1$.

This will be obtained in terms of the following quantities, which involve a more detailed account of the asymptotic behavior of the nonlinear term at infinity and as $\lambda \to \sigma_1$:

$$\underline{\mathbf{F}}_+ := \int_{\partial\Omega} \liminf_{(\lambda,s)\to(\sigma_1,+\infty)} \frac{sg(\lambda,\cdot,s) - s^2 g_u(\lambda,\cdot,s)}{|s|^{1+\alpha}} \Phi_1^{1+\alpha}$$

and

$$\overline{\mathbf{F}}_+ := \int_{\partial\Omega} \limsup_{(\lambda,s)\to(\sigma_1,+\infty)} \frac{sg(\lambda,\cdot,s) - s^2 g_u(\lambda,\cdot,s)}{|s|^{1+\alpha}} \Phi_1^{1+\alpha},$$

for some $\alpha < 1$. In this paper we show that, if $\underline{\mathbf{F}}_+ > 0$, any positive large solution is stable, subcritical and unique for each λ in a neighborhood of σ_1, see Theorem 3.4. On the other hand, if $\overline{\mathbf{F}}_+ < 0$, any positive large solution is unstable and supercritical and unique in a neighborhood of σ_1, see Theorem 3.5.

For example if

$$g(x,s) := a(x)s^{\alpha}, \qquad \text{for} \quad s \gg 1,$$

and $a(x)$ is such that $\int_{\partial\Omega} a\Phi_1^{1+\alpha} > 0$, then $\underline{\mathbf{F}}_+ > 0$. If, on the contrary, $\int_{\partial\Omega} a\Phi_1^{1+\alpha} < 0$, then $\overline{\mathbf{F}}_+ < 0$.

For the analysis in this paper, we need to consider several eigenvalue problems. If $b \in L^r(\partial\Omega)$, $r > N-1$, we denote by $\mu_1(b)$ and $\varphi_1 = \varphi_1(b) > 0$ the first Steklov eigenvalue and eigenfunction of the problem

$$\begin{cases} -\Delta\varphi_1 + \varphi_1 &= 0, & \text{in } \Omega \\ \frac{\partial\varphi_1}{\partial n} + b(x)\varphi_1 &= \mu_1(b)\varphi_1, & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

Also, we will denote by $\Lambda_1(b)$ and $\xi_1 = \xi_1(b) > 0$ the first eigenvalue and eigenfunction respectively of the following problem

$$\begin{cases} -\Delta\xi_1 + \xi_1 &= \Lambda_1\xi_1, & \text{in } \Omega \\ \frac{\partial\xi_1}{\partial n} + b(x)\xi_1 &= 0, & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

From maximum principles, it is well known that if $b_1 \le b_2$, $b_1 \ne b_2$, then $\mu_1(b_1) < \mu_1(b_2)$ and $\Lambda_1(b_1) < \Lambda_1(b_2)$.

Note also that for both (2.4) and (2.5), the first eigenvalue is simple and is the only one with a positive associated eigenfunction.

We will refer to Λ_1 in (2.5) as the *interior* eigenvalue, to distinguish it clearly from the *boundary* Steklov eigenvalue, μ_1 in (2.4). We will keep this notation on eigenvalues and eigenfunctions throughout the paper. Also, the first eigenfunction will be normalized in $L^\infty(\partial\Omega)$, unless otherwise stated.

In this paper, we also state sufficient conditions for monotonicity with respect to the parameter, of the large solutions of (1.2). This property will be obtained as consequence of a uniform antimaximum principle, see the Appendix, applied to the derivative of the solution with respect to the parameter λ, see Theorems 4.1 and 4.2. Roughly speaking, if

$$\frac{1}{s}\frac{\partial g}{\partial\lambda}(\lambda,x,s) \le C < 1, \text{ as } |s| \to \infty,$$

then any unbounded branch, either stable or unstable, is monotone with respect to the parameter, see Theorems 4.1 and 4.2. We note that this condition is satisfied whenever $g(\lambda,x,s) = \lambda g_0(x,s)$ and $g_0(x,s)$ is sublinear at infinity.

Concerning the parabolic problem (1.1), since g is locally Lipschitz in u uniformly in $x \in \partial\Omega$, then for each initial condition $u_0 \in C(\bar\Omega)$ we have a unique solution $u \in C([0,T], C(\bar\Omega))$. Moreover, (H1) and (H2) implies that g grows less than linear and therefore we have global existence of solutions, that is, we can take $T = +\infty$, see [4]. In particular, (1.1) defines a nonlinear semigroup of solutions that we denote $T_\lambda(t)$. From the regularity properties of the solutions we get that the semigroup is also compact, in the sense that if B is a set of initial data, bounded in $C(\bar\Omega)$, the evolution at time $t > 0$ of this set, $T_\lambda(t)B$ is bounded in $C^\alpha(\bar\Omega)$ and therefore compact in $C(\bar\Omega)$. Even more, if the set B is such that its orbit $\{T_\lambda(t)B,\ 0 \le t < \infty\}$ is bounded in $C(\bar\Omega)$, then it is actualy relatively compact in the same space.

Furthermore, the semigroup is order preserving and if $g(\lambda,x,0) \ge 0$ then the sign of a nonnegative initial data is preserved.

Additionaly, the semigroup generated by (1.1) is a gradient system. As a matter of fact, the function

$$E(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + |u|^2) - \int_{\partial\Omega} (\lambda u^2 + H(\lambda, x, u)),$$

where $H(\lambda, x, u) = \int_0^u g(\lambda, x, s)ds$, is a Lyapunov function for the system. This means that the omega limit sets of a solution of (1.1) which is bounded in $H^1(\Omega)$, is made up of equilibria, i.e. solutions of (1.2).

As mentioned before, if $\lambda < \sigma_1$ then, the flow defined by (1.1) is dissipative and compact, hence it will have a global attractor, see [6, Section 7]. On the other hand, when $\lambda > \sigma_1$ the flow is not longer dissipative and we have initial conditions for which the solution of (1.1) grows without bounds (blows-up in infinite time).

3 Stability or instability of positive equilibria bifurcating from infinity.

We analyze in this section the stability properties of the branches of solutions of (1.2) bifurcating from infinity at the first Steklov eigenvalue σ_1.

We sketch now the main argument that will lead to the stability and instability result. Let us denote by $u_\lambda > 0$ a solution of (1.2) bifurcating from infinity for λ near σ_1. The eigenvalue problem associated to the linearization around u_λ, as an equilibrium of (1.1), is given by

$$\begin{cases} -\Delta\xi + \xi &= \Lambda\xi, & \text{in } \Omega \\ \frac{\partial\xi}{\partial n} &= \lambda\xi + g_u(\lambda, x, u_\lambda)\xi, & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

where $g_u = \dfrac{\partial g}{\partial u}$. Thus the stability properties of u_λ are determined by the sign of the first eigenvalue of (3.1). Following the notations in (2.5), the eigenvalue can be written as $\Lambda_1 := \Lambda_1(-\lambda - g_u(\lambda, x, u_\lambda))$.

Let us also consider the auxiliary Steklov eigenvalue problem associated to the linearization around u_λ given by

$$\begin{cases} -\Delta\varphi + \varphi &= 0, & \text{in } \Omega \\ \frac{\partial\varphi}{\partial n} &= \mu\varphi + g_u(\lambda, x, u_\lambda)\varphi, & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Observe that with the notations of (2.4), the first eigenvalue of (3.2) can be written as $\mu_1 := \mu_1(-g_u(\lambda, \cdot, u_\lambda))$.

Now we use that for both eigenvalue problems (3.1), (3.2) the first eigenvalue is the only one with a positive eigenfunction. This implies that in (3.2) the first *interior* eigenvalue associated to the boundary potential $b(x) = -\mu_1 - g_u(\lambda, x, u_\lambda)$ satisfies $\Lambda_1(-\mu_1 - g_u(\lambda, x, u_\lambda)) = 0$, while in (3.1) the first eigenvalue is $\Lambda_1(-\lambda - g_u(\lambda, x, u_\lambda))$. Hence, if we are able to compare μ_1 in (3.2) with λ, then in (3.1) we will have that u_λ is stable if $\mu_1 > \lambda$ and unstable if $\mu_1 < \lambda$.

Therefore, we need to figure out a tool to compare μ_1 with λ, as $\lambda \to \sigma_1$. This will be achieved in Lemma 3.3 below. For this we look at the lower order terms of $g(\lambda, x, s)$ as $\lambda \to \sigma_1$ and $s \to \infty$. Hence, we define, for $\alpha < 1$, the following quantities

$$\underline{\mathbf{G}}_+ \; := \; \int_{\partial\Omega} \liminf_{\substack{\lambda\to\sigma_1 \\ s\to+\infty}} \frac{sg(\lambda, x, s)}{|s|^{1+\alpha}} \; \Phi_1^{1+\alpha},$$

$$
\begin{aligned}
\underline{\mathbf{D}}_+ \quad &:= \quad \int_{\partial\Omega} \liminf_{\substack{\lambda\to\sigma_1 \\ s\to+\infty}} \frac{g_u(\lambda,x,s)}{|s|^{\alpha-1}}\ \Phi_1^{1+\alpha}, \\
\underline{\mathbf{F}}_+ \quad &:= \quad \int_{\partial\Omega} \liminf_{\substack{\lambda\to\sigma_1 \\ s\to+\infty}} \frac{sg(\lambda,x,s)-s^2 g_u(\lambda,x,s)}{|s|^{1+\alpha}}\ \Phi_1^{1+\alpha},
\end{aligned} \tag{3.3}
$$

where Φ_1 is the first Steklov eigenfunction as in (1.3) with $\|\Phi_1\|_{L^\infty(\partial\Omega)} = 1$. Changing $\liminf$ by $\limsup$ we define the numbers $\overline{\mathbf{G}}_+$, $\overline{\mathbf{D}}_+$, $\overline{\mathbf{F}}_+$ and considering the limits when $s \to -\infty$ we will have defined $\underline{\mathbf{G}}_-$, $\underline{\mathbf{D}}_-$, $\underline{\mathbf{F}}_-$ and $\overline{\mathbf{G}}_-$, $\overline{\mathbf{D}}_-$, $\overline{\mathbf{F}}_-$.

Note that $\underline{\mathbf{G}}_+$, $\overline{\mathbf{G}}_+$, $\underline{\mathbf{G}}_-$ and $\overline{\mathbf{G}}_-$ where used in [6] to determine the subcritical or supercritical nature of the bifurcation at σ_1.

Also, observe that the difficulty of comparing μ_1 and λ is that, as $\lambda \to \sigma_1$ we have $\mu_1 \to \sigma_1$ as well, see Lemma 3.2 below.

Let us consider now two technical lemmas that will be the key to prove Lemma 3.3. The first one is basically a restatement of [6, Lemma 4.2] and it was used to determine whether the bifurcation is subcritical or supercritical. Note that this result allows us to compare σ_1 and λ.

Lemma 3.1 *Assume the nonlinearity g satisfies hypotheses (H1) and (H2). Denote by σ_1 the first Steklov eigenvalue and by Φ_1 the first positive eigenfunction with $\|\Phi_1\|_{L^\infty(\partial\Omega)} = 1$ as in (1.3). Assume that for some $\alpha < 1$ there exists a function G_1 such that for $\lambda \to \sigma_1$, for sufficiently large $s > 0$ and $x \in \partial\Omega$ we have*

$$
\frac{|g(\lambda,x,s)|}{|s|^\alpha} \le G_1(x), \quad G_1 \in L^1(\partial\Omega)
$$

Consider (λ_n, u_n), a sequence of solutions of (1.2) such that $\lambda_n \to \sigma_1$ and $\|u_n\|_{L^\infty(\partial\Omega)} \to \infty$. Then, if $u_n > 0$ we have

$$
\begin{aligned}
\frac{\underline{\mathbf{G}}_+}{\int_{\partial\Omega} \Phi_1{}^2} \quad &\le \quad \frac{1}{\int_{\partial\Omega} \Phi_1{}^2}\ \liminf_{n\to\infty} \int_{\partial\Omega} \frac{u_n g(\lambda_n,\cdot,u_n)}{|u_n|^{1+\alpha}} \Phi_1^{1+\alpha} \le \liminf_{n\to\infty} \frac{\sigma_1-\lambda_n}{\|u_n\|_{L^\infty(\partial\Omega)}^{\alpha-1}} \\
&\le \quad \limsup_{n\to\infty} \frac{\sigma_1-\lambda_n}{\|u_n\|_{L^\infty(\partial\Omega)}^{\alpha-1}} \le \frac{1}{\int_{\partial\Omega} \Phi_1{}^2}\ \limsup_{n\to\infty} \int_{\partial\Omega} \frac{u_n g(\lambda_n,\cdot,u_n)}{|u_n|^{1+\alpha}} \Phi_1^{1+\alpha} \le \frac{\overline{\mathbf{G}}_+}{\int_{\partial\Omega} \Phi_1{}^2}.
\end{aligned}
$$

A similar statement is obtained for the case $u_n < 0$, just changing $\underline{\mathbf{G}}_+$ by $\underline{\mathbf{G}}_-$ and $\overline{\mathbf{G}}_+$ by $\overline{\mathbf{G}}_-$.

Proof. See [6, Lemma 4.2]. □

Let us now denote by $u_\lambda > 0$ a solution of (1.2) bifurcating from infinity. We consider the auxiliary linearized Steklov eigenvalue problem (3.2) and, with the notations in (2.4), denote the first eigenvalue by $\mu_1 = \mu_1(-g_u(\lambda,\cdot,u_\lambda))$ and the first positive eigenfunction by $\varphi_1 = \varphi_1(\lambda, u_\lambda)$, which we assume normalized in $L^\infty(\partial\Omega)$ so that $\|\varphi_1\|_{L^\infty(\partial\Omega)} = 1$.

The next result states sufficient condition for the convergence of $\mu_1 \to \sigma_1$ and of $\varphi_1 \to \Phi_1$ as $\lambda \to \sigma_1$ and allows to compare μ_1 and σ_1.

Lemma 3.2 *Assume the nonlinearity g satisfies hypotheses (H1), (H2) and (H3).*

Assume that for some $\alpha < 1$ there exists a function D_1 such that for $\lambda \to \sigma_1$, for sufficiently large $s > 0$ and $x \in \partial\Omega$ we have

$$\frac{|g_u(\lambda, x, s)|}{|s|^{\alpha-1}} \leq D_1(x), \quad D_1 \in L^r(\partial\Omega) \quad \textit{with } r > N-1. \tag{3.4}$$

Then the first eigenvalue and eigenfunction in (3.2) satisfy

$$\mu_1(-g_u(\lambda, \cdot, u_\lambda)) \to \sigma_1 \quad \textit{as} \quad \lambda \to \sigma_1 \tag{3.5}$$

$$\varphi_1(\lambda, u_\lambda) \to \Phi_1 \quad \textit{in } H^1(\Omega) \cap C^\beta(\overline{\Omega}) \quad \textit{as} \quad \lambda \to \sigma_1 \tag{3.6}$$

for some $\beta \in (0,1)$.

Moreover for any sequence of solutions of (1.2), (λ_n, u_n) such that $\lambda_n \to \sigma_1$ and $\|u_n\|_{L^\infty(\partial\Omega)} \to \infty$, setting $\mu_{1,n} = \mu_1(-g_u(\lambda_n, \cdot, u_n))$, we have, if $u_n > 0$

$$\frac{\underline{\mathbf{D}}_+}{\int_{\partial\Omega} {\Phi_1}^2} \leq \liminf_{n\to\infty} \frac{\sigma_1 - \mu_{1,n}}{\|u_n\|^{\alpha-1}_{L^\infty(\partial\Omega)}} \leq \limsup_{n\to\infty} \frac{\sigma_1 - \mu_{1,n}}{\|u_n\|^{\alpha-1}_{L^\infty(\partial\Omega)}} \leq \frac{\overline{\mathbf{D}}_+}{\int_{\partial\Omega} {\Phi_1}^2}. \tag{3.7}$$

A similar statement is obtained for the case $u_n < 0$, just changing $\underline{\mathbf{D}}_+$ by $\underline{\mathbf{D}}_-$ and $\overline{\mathbf{D}}_+$ by $\overline{\mathbf{D}}_-$.

Proof. Note that, using $\alpha < 1$, (3.4) and (1.4), in (3.2) the boundary potential satisfies

$$g_u(\lambda, \cdot, u_\lambda) = \|u_\lambda\|^{\alpha-1}_{L^\infty(\partial\Omega)} \frac{g_u(\lambda, \cdot, u_\lambda)}{|u_\lambda|^{\alpha-1}} \left(\frac{|u_\lambda|}{\|u_\lambda\|_{L^\infty(\partial\Omega)}}\right)^{\alpha-1} \to 0 \quad \text{in } L^r(\partial\Omega),$$

as $\lambda \to \sigma_1$.

From this, the spectrum of the linear operator also passes to the limit since $r > N-1$ and then $\varphi_1(\lambda, u_\lambda) \to \Phi_1$ in $H^1(\Omega)$ as $\lambda \to \sigma_1$, see Proposition A.2 in the Appendix. The elliptic regularity imply now that (3.6) is satisfied.

On the other hand, if $u_n > 0$, considering equation (3.2) for the first eigenfunction, multiplying it by Φ_1 and integrating by parts, we get

$$(\sigma_1 - \mu_{1,n}) \int_{\partial\Omega} \varphi_{1,n}\Phi_1 = \int_{\partial\Omega} g_u(\lambda_n, \cdot, u_n)\varphi_{1,n}\Phi_1 \tag{3.8}$$

where $\varphi_{1,n} = \varphi_1(\lambda_n, u_n)$. But,

$$\int_{\partial\Omega} g_u(\lambda_n, \cdot, u_n)\varphi_{1,n}\Phi_1 = \|u_n\|^{\alpha-1}_{L^\infty(\partial\Omega)} \int_{\partial\Omega} \frac{g_u(\lambda_n, \cdot, u_n)}{|u_n|^{\alpha-1}} \left(\frac{|u_n|}{\|u_n\|_{L^\infty(\partial\Omega)}}\right)^{\alpha-1} \varphi_{1,n}\Phi_1$$

and

$$\begin{aligned}
\liminf_{n\to\infty} & \int_{\partial\Omega} \frac{g_u(\lambda_n, \cdot, u_n)}{|u_n|^{\alpha-1}} \left(\frac{|u_n|}{\|u_n\|_{L^\infty(\partial\Omega)}}\right)^{\alpha-1} \varphi_{1,n}\Phi_1 \\
\geq & \liminf_{n\to\infty} \int_{\partial\Omega} \frac{g_u(\lambda_n, \cdot, u_n)}{|u_n|^{\alpha-1}} \left[\left(\frac{|u_n|}{\|u_n\|_{L^\infty(\partial\Omega)}}\right)^{\alpha-1} - \Phi_1^{\alpha-1}\right] \varphi_{1,n}\Phi_1
\end{aligned}$$

$$
\begin{aligned}
& + \liminf_{n\to\infty} \int_{\partial\Omega} \frac{g_u(\lambda_n,\cdot,u_n)}{|u_n|^{\alpha-1}} \varphi_{1,n}\ \Phi_1^\alpha \\
\geq \quad & \liminf_{n\to\infty} \int_{\partial\Omega} \frac{g_u(\lambda_n,\cdot,u_n)}{|u_n|^{\alpha-1}} [\varphi_{1,n} - \Phi_1]\ \Phi_1^\alpha \\
& + \int_{\partial\Omega} \liminf_{n\to\infty} \frac{g_u(\lambda_n,\cdot,u_n)}{|u_n|^{\alpha-1}} \Phi_1^{1+\alpha} \geq \underline{\mathbf{D}}_+
\end{aligned}
$$

where we have used again that $\Phi_1 > 0$ for all x on $\partial\Omega$, (1.4), (3.6) and Fatou's Lemma.

Dividing in (3.8) by $\|u_n\|^{\alpha-1}_{L^\infty(\partial\Omega)}$ and passing to the limit we obtain the first inequality of (3.7). The second inequality is obvious and the third one is obtained similarly to the first one.$\square$

We are now in a position to prove the following result, from which stability and instability will be derived. Note that this result allows us to compare λ and μ_1 as $\lambda \to \sigma_1$.

Lemma 3.3 *Assume the hypotheses of Lemma 3.2 hold. Assume that for some $\alpha < 1$ there exists a function F_1 such that for $\lambda \to \sigma_1$, for sufficiently large $s > 0$ and $x \in \partial\Omega$ we have*

$$
\frac{|g(\lambda,x,s) - s g_u(\lambda,x,s)|}{|s|^\alpha} \leq F_1(x), \quad F_1 \in L^1(\partial\Omega) \tag{3.9}
$$

then for any sequence of solutions of (1.2) (λ_n,u_n) such that $\lambda_n \to \sigma_1$ and $\|u_n\|_{L^\infty(\partial\Omega)} \to \infty$ denoting by $\mu_{1,n} = \mu_1(-g_u(\lambda_n,\cdot,u_n))$, the first eigenvalue in (3.2), we have, if $u_n > 0$

$$
\begin{aligned}
\frac{\underline{\mathbf{F}}_+}{\int_{\partial\Omega} {\Phi_1}^2} \quad & \leq \quad \frac{1}{\int_{\partial\Omega} {\Phi_1}^2} \liminf_{n\to\infty} \int_{\partial\Omega} \frac{u_n g(\lambda_n,x,u_n) - u_n^2 g_u(\lambda_n,x,u_n)}{|u_n|^{1+\alpha}} \Phi_1^{1+\alpha} \\
& \leq \quad \liminf_{n\to\infty} \frac{\mu_{1,n} - \lambda_n}{\|u_n\|^{\alpha-1}_{L^\infty(\partial\Omega)}} \leq \limsup_{n\to\infty} \frac{\mu_{1,n} - \lambda_n}{\|u_n\|^{\alpha-1}_{L^\infty(\partial\Omega)}} \\
& \leq \quad \frac{1}{\int_{\partial\Omega} {\Phi_1}^2} \limsup_{n\to\infty} \int_{\partial\Omega} \frac{u_n g(\lambda_n,x,u_n) - u_n^2 g_u(\lambda_n,x,u_n)}{|u_n|^{1+\alpha}} \Phi_1^{1+\alpha} \leq \frac{\overline{\mathbf{F}}_+}{\int_{\partial\Omega} {\Phi_1}^2}
\end{aligned}
$$

A similar statement is obtained for the case $u_n < 0$, just changing $\underline{\mathbf{F}}_+$ by $\underline{\mathbf{F}}_-$ and $\overline{\mathbf{F}}_+$ by $\overline{\mathbf{F}}_-$.

Proof. Taking u_n as the test function in the variational formulation of the first eigenfunction in (3.2), we have

$$
(\mu_1 - \lambda_n) \int_{\partial\Omega} u_n \varphi_{1,n} = \int_{\partial\Omega} [g(\lambda_n,\cdot,u_n) - g_u(\lambda_n,\cdot,u_n)u_n]\varphi_{1,n},
$$

with $\varphi_{1,n} = \varphi_1(\lambda_n,u_n)$. Now,

$$
\frac{\int_{\partial\Omega} [g(\lambda_n,\cdot,u_n) - g_u(\lambda_n,\cdot,u_n)u_n]\varphi_{1,n}}{\|u_n\|^\alpha_{L^\infty(\partial\Omega)}} = \int_{\partial\Omega} \frac{g(\lambda_n,\cdot,u_n) - g_u(\lambda_n,\cdot,u_n)u_n}{|u_n|^\alpha} \left(\frac{|u_n|}{\|u_n\|_{L^\infty(\partial\Omega)}}\right)^\alpha \varphi_{1,n}.
$$

Let us observe that from the hypothesis (3.9), using that $\Phi_1 > 0$ for all x on $\partial\Omega$ and (1.4), we obtain

$$\int_{\partial\Omega} \left| \frac{g(\lambda_n, \cdot, u_n) - g_u(\lambda_n, \cdot, u_n)u_n}{|u_n|^\alpha} \left[\left(\frac{|u_n|}{\|u_n\|_{L^\infty(\partial\Omega)}} \right)^\alpha - \Phi_1^\alpha \right] \varphi_{1,n} \right| \leq$$

$$\leq C \left\| \left(\frac{|u_n|}{\|u_n\|_{L^\infty(\partial\Omega)}} \right)^\alpha - \Phi_1^\alpha \right\|_{L^\infty(\partial\Omega)} \to 0, \text{ as } \lambda_n \to \sigma_1.$$

From (3.6) and hypothesis (3.9), we get

$$\int_{\partial\Omega} \left| \frac{g(\lambda_n, \cdot, u_n) - g_u(\lambda_n, \cdot, u_n)u_n}{|u_n|^\alpha} \right| \Phi_1^\alpha \, |\varphi_{1,n} - \Phi_1| \leq C \|\varphi_{1,n} - \Phi_1\|_{L^\infty(\partial\Omega)} \to 0, \text{ as } \lambda_n \to \sigma_1.$$

Moreover, using Fatou's Lemma and the definition of $\underline{\mathbf{F}}_+$, we can write

$$\begin{aligned}
&\liminf_{n\to\infty} \int_{\partial\Omega} \frac{g(\lambda_n, \cdot, u_n) - g_u(\lambda_n, \cdot, u_n)u_n}{|u_n|^\alpha} \left(\frac{u_n}{\|u_n\|_{L^\infty(\partial\Omega)}} \right)^\alpha \varphi_{1,n} \\
&\geq \lim_{n\to\infty} \int_{\partial\Omega} \frac{g(\lambda_n, \cdot, u_n) - g_u(\lambda_n, \cdot, u_n)u_n}{|u_n|^\alpha} \left[\left(\frac{u_n}{\|u_n\|_{L^\infty(\partial\Omega)}} \right)^\alpha - \Phi_1^\alpha \right] \varphi_{1,n} \\
&\quad + \lim_{n\to\infty} \int_{\partial\Omega} \frac{g(\lambda_n, \cdot, u_n) - g_u(\lambda_n, \cdot, u_n)u_n}{|u_n|^\alpha} \Phi_1^\alpha \left(\varphi_{1,n} - \Phi_1 \right) \\
&\quad + \liminf_{n\to\infty} \int_{\partial\Omega} \frac{g(\lambda_n, \cdot, u_n) - g_u(\lambda_n, \cdot, u_n)u_n}{|u_n|^\alpha} \Phi_1^{1+\alpha} \\
&\geq \int_{\partial\Omega} \liminf_{n\to\infty} \frac{g(\lambda_n, \cdot, u_n) - g_u(\lambda_n, \cdot, u_n)u_n}{|u_n|^\alpha} \Phi_1^{1+\alpha} \geq \underline{\mathbf{F}}_+.
\end{aligned}$$

The other inequality is obtained in a similar way. This concludes the proof of the lemma.□

With this result, we can proceed now to analyze the stability properties of the solutions of (1.2) bifurcating from infinity. The first result provides sufficient conditions for the stability of positive solutions of (1.2) bifurcating from infinity. It also states that, under those hypotheses, the stable branch is subcritical and unique in a neighborhood of σ_1. In other words, as $\lambda \to \sigma_1$ the branch of unbounded positive solutions of (1.2) is composed of stable subcritical solutions and u_λ is unique for each λ.

Note that in [6] a preliminary result was proved in Proposition 7.1.

Theorem 3.4 (Stability for subcritical equilibria bifurcating from infinity). *Assume the nonlinearity g satisfies hypotheses (H1), (H2) and (H3). Assume that for some $\alpha < 1$ there exist functions D_1, F_1 such that for $\lambda \to \sigma_1$, for sufficiently large $s > 0$ and $x \in \partial\Omega$ we have*

$$\frac{|g_u(\lambda, x, s)|}{|s|^{\alpha-1}} \leq D_1(x), \quad D_1 \in L^r(\partial\Omega)$$

$$\frac{|g(\lambda, x, s) - s g_u(\lambda, x, s)|}{|s|^\alpha} \leq F_1(x), \quad F_1 \in L^1(\partial\Omega)$$

Assume also the following condition holds

$$\underline{\mathbf{F}}_+ \; > 0. \tag{3.10}$$

Then, for λ in a neighborhood of σ_1 the following assertions hold.

i) *The bifurcation from infinity of positive solutions of (1.2) at $\lambda = \sigma_1$ is subcritical,*

ii) *The positive solution of (1.2) in the branch bifurcating from infinity is unique for each fixed $\lambda \approx \sigma_1$. That is, there exists a small $\delta > 0$ and a large number $M > 0$ such that for each $\sigma_1 - \delta < \lambda < \sigma_1$, there is a unique positive solution of (1.2) u_λ with $\|u_\lambda\|_{L^\infty(\partial\Omega)} \geq M$.*

Even more, this solution is asymptotically stable and its basin of attraction includes all initial conditions which are large enough, i.e. satisfying $\|u_0\|_{L^\infty(\partial\Omega)} \geq M$, with M large enough and uniform for all $\sigma_1 - \delta < \lambda < \sigma_1$.

An analogous result holds for negative solutions under the assumption $\underline{\mathbf{F}}_- > 0$.

Proof. We first prove that $\underline{\mathbf{F}}_+ > 0$ implies $\underline{\mathbf{G}}_+ > 0$ which implies that the bifurcation is subcritical, see Theorem 4.3 in [6]. Let us consider $\varepsilon > 0$ a small number. Now, for $x \in \partial\Omega$ fixed, we have

$$\frac{\partial}{\partial s}\left[\frac{g(\lambda, x, s)}{s}\right] = -\frac{g(\lambda, x, s) - s g_u(\lambda, x, s)}{s^2}$$

and if we define

$$\underline{F}_+(x) := \liminf_{\substack{\lambda \to \sigma_1 \\ s \to +\infty}} \frac{s g(\lambda, x, s) - s^2 g_u(\lambda, x, s)}{|s|^{1+\alpha}}$$

we will have that, as $\lambda \to \sigma_1$, for sufficiently large $s > 0$ and $x \in \partial\Omega$

$$\frac{\partial}{\partial s}\left[\frac{g(\lambda, x, s)}{s}\right] \leq -s^{\alpha-2}[\underline{F}_+(x) - \varepsilon].$$

Integrating now from s to s_1 we deduce

$$\frac{g(\lambda, x, s_1)}{s_1} - \frac{g(\lambda, x, s)}{s} \leq \frac{\underline{F}_+(x) - \varepsilon}{1-\alpha}\left(s_1^{\alpha-1} - s^{\alpha-1}\right).$$

Letting $s_1 \to \infty$ for fixed $x \in \partial\Omega$, we have $\frac{g(\lambda,x,s_1)}{s_1} \to 0$ and then

$$\frac{g(\lambda, x, s)}{s^\alpha} \geq \frac{\underline{F}_+(x) - \varepsilon}{1-\alpha}.$$

Passing to the limit as $\lambda \to \sigma_1$ and $s \to \infty$, we get

$$\liminf_{\substack{\lambda \to \sigma_1 \\ s \to +\infty}} \frac{s g(\lambda, x, s)}{|s|^{1+\alpha}} \geq \frac{\underline{F}_+(x) - \varepsilon}{1-\alpha}, \qquad \forall x \in \partial\Omega. \tag{3.11}$$

Moreover, since (3.11) is valid for all $\varepsilon > 0$ arbitrarily small, we will have

$$\liminf_{\substack{\lambda \to \sigma_1 \\ s \to +\infty}} \frac{s g(\lambda, x, s)}{|s|^{1+\alpha}} \geq \frac{\underline{F}_+(x)}{1-\alpha}, \qquad \forall x \in \partial\Omega$$

Multiplying by $\Phi_1^{1+\alpha}$ and integrating on $\partial\Omega$ we obtain, from (3.3)

$$\underline{\mathbf{G}}_+ \geq \frac{\underline{\mathbf{F}}_+}{1-\alpha} > 0.$$

Let us now prove that any positive solution of (1.2) bifurcating from infinity is stable. For this we follow the argument sketched at the beginning of this Section. Let us denote by $u_\lambda > 0$ a solution of (1.2) bifurcating from infinity. The eigenvalue problem associated to the linearization around u_λ, is given by (3.1) . Hence, we will show that the first eigenvalue is positive for λ close enough to σ_1. To do that we note that with the notations in (2.5) we have that the first eigenvalue of (3.1) can be written as $\Lambda_1 = \Lambda_1(-\lambda - g_u(\lambda, x, u_\lambda))$. Then we consider first eigenvalue μ_1 of the auxiliary Steklov linearized eigenvalue problem (3.2). Then, in (3.2), the notations in (2.5) imply that the first *interior* eigenvalue satisfies $\Lambda_1(-\mu_1 - g_u(\lambda, x, u_\lambda)) = 0$. As we show below that $\mu_1 > \lambda$, we get then $\Lambda_1 = \Lambda_1(-\lambda - g_u(\lambda, x, u_\lambda)) > 0$ and obtain the stability. Hence, to conclude the proof note that using Lemma 3.3 and the hypothesis (3.10) we have

$$\liminf_{\lambda\to\sigma_1} \frac{\mu_1 - \lambda}{\|u_\lambda\|^{\alpha-1}_{L^\infty(\partial\Omega)}} \;\geq\; \frac{\underline{\mathbf{F}}_+}{\int_{\partial\Omega} {\Phi_1}^2} \;>\; 0.$$

and therefore $\mu_1 > \lambda$ for λ close enough to σ_1.

We will now prove uniqueness of large solutions of (1.2) for fixed λ close to σ_1. From the previous results there exists a $\delta > 0$ small enough and $M > 0$ large enough such that for $\lambda \in (\sigma_1 - \delta, \sigma_1)$, there exists at least one solution of (1.2) with $u_\lambda > 0$ and $\|u_\lambda\|_{L^\infty(\partial\Omega)} \geq M$ and also any such solution is asymptotically stable. Moreover, from (1.4) and (1.5), and maybe choosing a smaller $\delta > 0$, we have that any positive solution of (1.2) u bifurcating from infinity actually satisfies $u(x) > M$ for all $x \in \bar{\Omega}$. Let us denote by $\mathcal{E}_\lambda$ the set of solutions of (1.2) satisfying $u(x) > M$ for all $x \in \bar{\Omega}$. Our objective is to show that $\mathcal{E}_\lambda$ is a singleton.

Since all solutions in $\mathcal{E}_\lambda$ are asymptotically stable, we will have only a finite number of them. Moreover, applying [6, Proposition 7.1], we will have that for fixed $\lambda \in (\sigma_1 - \delta, \sigma_1)$ there exists a maximal solution in $\mathcal{E}_\lambda$, that is, there exists $u_\lambda \in \mathcal{E}_\lambda$ such that for any other $v \in \mathcal{E}_\lambda$ we have $v \leq u_\lambda$.

Let us assume that there exists $v_0 \in \mathcal{E}_\lambda$ with $v_0 \neq u_\lambda$. By the strong maximum principle, we will have that $v_0(x) < u_\lambda(x)$ for all $x \in \bar{\Omega}$. Moreover, if we define the set $[v_0, u_\lambda] = \{\varphi \in C(\bar{\Omega}), v_0(x) \leq \varphi(x) \leq u_\lambda(x)\}$ we will have that this set is positively invariant under the flow defined by (1.1), $T_\lambda(t)$. That is, if $T_\lambda(t)\varphi$ denotes the solution of (1.1) with initial condition φ and if $\varphi \in [v_0, u_\lambda]$ then $T_\lambda(t)\varphi \in [v_0, u_\lambda]$ for all $t > 0$.

Since $T_\lambda(t)$ is a gradient system, see Section 2, then $T_\lambda(t)\varphi$ must converge to one of the equilibriums in the interval $[v_0, u_\lambda]$ which we denote $\{v_0, v_1, \dots, v_{k+1} = u_\lambda\}$.

Let us consider now the convex linear combination of the functions v_0 and u_λ, that is, $\varphi_\eta = (1-\eta)v_0 + \eta v_{k+1} \in [v_0, u_\lambda]$ for $\eta \in [0,1]$. Define the function $h : [0,1] \to \{0, 1, \dots, k+1\}$ as follows: $h(\eta) = j$ if $T_\lambda(t)\varphi_\eta \to v_j$ as $t \to +\infty$. Observe that this function is well defined and that we have $h(0) = 0$ and $h(1) = k+1$. Moreover since all equilibria are asymptotically stable and using the continuous dependence of the solutions of (1.1) with respect to initial conditions in finite intervals of time, we can easily show that h is continuous. Hence, it is a constant function, which is impossible since $h(0) = 0$ and $h(1) = k+1$. Therefore, there cannot exist a function v_0 in $\mathcal{E}_\lambda$ different from u_λ.

The global asymptotic stability (with respect to large solutions of (1.1)) of the unique positive large equilibrium of (1.2) follows as in the proof of Proposition 7.1 in [6]. □

We state now a result on the instability of solutions for the case of a supercritical bifurcation. Now this result provides sufficient conditions for the instability of positive solutions of (1.2) bifurcating from infinity. It also states that, under those hypotheses, the unstable branch is supercritical and unique in a neighborhood of σ_1. In other words, as $\lambda \to \sigma_1$ the unbounded branch of positive solutions of (1.2) is composed of unstable supercritical solutions and u_λ is unique for each λ.

Note that in Proposition 7.3 in [6] a preliminary result was obtained.

Theorem 3.5 (Instability for supercritical equilibria bifurcating from infinity).
Assume the nonlinearity g satisfies hypotheses (H1), (H2) and (H3), see (2.1), (2.2), (2.3). Assume for some $\alpha < 1$ there exist functions D_1, F_1 such that such that for $\lambda \to \sigma_1$, for sufficiently large $s > 0$ and $x \in \partial\Omega$ we have

$$\frac{|g_u(\lambda, x, s)|}{|s|^{\alpha-1}} \leq D_1(x), \quad D_1 \in L^r(\partial\Omega) \quad \textit{with } r > N-1 \tag{3.12}$$

$$\frac{|g(\lambda, x, s) - s g_u(\lambda, x, s)|}{|s|^{\alpha}} \leq F_1(x), \quad F_1 \in L^1(\partial\Omega).$$

Assume also the following condition holds

$$\overline{\mathbf{F}}_+ \ < 0. \tag{3.13}$$

Then, for λ in a neighborhood of σ_1 the following assertions hold.

i) *The bifurcation from infinity of positive solutions of (1.2) at $\lambda = \sigma_1$ is supercritical.*

ii) *The positive equilibrium solution of (1.2) contained in the branch bifurcating from infinity is unique for each λ close enough to σ_1 and it is unstable.*

An analogous result holds for negative solutions of (1.2) under the assumption $\overline{\mathbf{F}}_- \ < 0$.

Proof. To prove that the bifurcation is supercritical we proceed as in the proof of Theorem 3.4. We therefore skip the details here.

To prove the instability, we proceed as in the proof of Theorem 3.4, but now from Lemma 3.3 we have

$$\limsup_{\lambda \to \sigma_1} \frac{\mu_1 - \lambda}{\|u_\lambda\|^{\alpha-1}_{L^\infty(\partial\Omega)}} \leq \frac{\overline{\mathbf{F}}_+}{\int_{\partial\Omega} {\Phi_1}^2} < 0.$$

and therefore $\mu_1 < \lambda$ for λ close enough to σ_1 and the equilibrium is unstable.

Now we prove the uniqueness of the solution in the branch. Assume on the contrary that for some sequence $\lambda_n \to \sigma_1$, with $\lambda_n > \sigma_1$ there exist two different supercritical unstable positive solutions of (1.2), u_n and v_n, satisfying (1.4).

Note then that u_n and v_n can not be ordered, since otherwise, there would be a stable large solution in between. This would contradict the instability shown above. Let us define

$w_n = u_n - v_n$, w_n which changes sign in Ω. By substracting the equations satisfied by u_n and v_n and taking Φ_1 as a test function, we get

$$(\sigma_1 - \lambda_n) \int_{\partial\Omega} w_n \Phi_1 = \int_{\partial\Omega} [g(\lambda_n, \cdot, u_n) - g(\lambda_n, \cdot, v_n)]\, \Phi_1. \tag{3.14}$$

Let us write

$$g(\lambda_n, x, u_n) - g(\lambda_n, x, v_n) = w_n \int_0^1 g_u(\lambda_n, x, \tau u_n + (1-\tau) v_n)\; d\tau,$$

and set $b_n(x) := \int_0^1 g_u(\lambda_n, x, \tau u_n + (1-\tau)v_n)\; d\tau$. Using (1.5) and (3.12) we can assert that

$$b_n \to 0 \text{ in } L^r(\partial\Omega), \text{ with } r > N-1. \tag{3.15}$$

Set now $z_n = \frac{w_n}{\|w_n\|_{L^\infty(\partial\Omega)}}$. Then z_n satisfies the following problem

$$\begin{cases} -\Delta z_n + z_n &= 0, & \text{in } \Omega \\ \dfrac{\partial z_n}{\partial n} &= \lambda_n z_n + b_n(x) z_n, & \text{on } \partial\Omega \end{cases}$$

with $\|z_n\|_{L^\infty(\partial\Omega)} = 1$.

From here, taking into account that $b_n \in L^r(\partial\Omega)$ for $r > N-1$, see (3.15), and using regularity results for the linear problem, see for instance [6, lemma 2.1], we then get $\|z_n\|_{C^\alpha(\overline{\Omega})} \leq C$ for some $\alpha \in (0,1)$. By the compact imbedding $C^\alpha(\overline{\Omega}) \hookrightarrow C^\beta(\overline{\Omega})$ for $0 < \beta < \alpha$ and taking subsequences if necessary, we can assume that z_n converges to z in $C^\beta(\overline{\Omega})$. Hence $\|z\|_{L^\infty(\partial\Omega)} = 1$. Moreover, using (3.15), z is an eigenfunction of the Steklov eigenvalue problem (1.3), associated to the first eigenvalue σ_1, see Proposition A.1 in the Appendix. Since this is simple, we deduce either $z > 0$ or $z < 0$ and in any case either $z_n > 0$ or $z_n < 0$ or equivalently either $w_n > 0$ or $w_n < 0$ which contradicts the fact that w_n changes sign. Therefore, for λ sufficiently close to σ_1 the solution of (1.2) bifurcating from infinity is unique. $\square$

4 Monotonicity with respect to the parameter

In this section we give sufficient conditions for the monotonicity with respect to the parameter of the unbounded branch. This property will be a consequence of the uniform antimaximum principle developed in the Appendix in this paper. We will apply this antimaximum principle to the derivative of the solution with respect to the parameter.

We make the following extra hypothesis:

(H4) The function g is differentiable with respect to the parameter λ and moreover there exists a function G_2 such that for $\lambda \to \sigma_1$, for sufficiently large $|s|$ and $x \in \partial\Omega$ we have

$$\left| \frac{g_\lambda(\lambda, x, s)}{s} \right| \leq G_2(x), \quad G_2 \in L^r(\partial\Omega) \tag{4.1}$$

and

$$\int_{\partial\Omega} \liminf_{(\lambda,|s|)\to(\sigma_1,\infty)} \left[1 + \frac{g_\lambda(\lambda, x, s)}{s} \right] \Phi_1^2 > 0, \tag{4.2}$$

where $g_\lambda = \dfrac{\partial g}{\partial \lambda}$ and $\Phi_1 > 0$ is the first Steklov eigenfunction of (1.3).

Observe that this condition is satisfied for any nonlinearity $g(\lambda, x, s) = \lambda g_0(x, s)$ satisfying $(H2)$, see (2.2), or whenever g is independent of λ.

The following result states that, under the conditions of Theorem 3.4 and assuming also $(H4)$ the subcritical branch of stable solutions is monotone.

Theorem 4.1 (Monotonicity for positive large equilibria)
i) Assume the conditions of Theorem 3.4 hold. If moreover (H4) is satisfied, then the unique subcritical branch of positive solutions of (1.2), u_λ, bifurcating from infinity as $\lambda \to \sigma_1$ is increasing with respect to λ, with λ close enough to σ_1. Even more

$$\frac{\partial u_\lambda}{\partial \lambda}(x) > 0 \qquad \textit{for all } x \in \Omega.$$

ii) Assume the conditions of Theorem 3.5 hold. If moreover (H4) is satisfied, then the unique supercritical branch of positive solutions of (1.2), u_λ, bifurcating from infintity as $\lambda \to \sigma_1$ is decreasing with respect to λ. Even more

$$\frac{\partial u_\lambda}{\partial \lambda}(x) < 0 \qquad \textit{for all } x \in \Omega.$$

Proof. Set $v := \dfrac{\partial u_\lambda}{\partial \lambda}$. Taking derivatives in (1.2) with respect to λ we obtain

$$\left\{ \begin{array}{rcll} -\Delta v + v &=& 0, & \text{in } \Omega \\ \dfrac{\partial v}{\partial n} &=& [\lambda + g_u(\lambda, x, u_\lambda)]v + u_\lambda + g_\lambda(\lambda, x, u_\lambda), & \text{on } \partial\Omega. \end{array} \right. \tag{4.3}$$

To achieve the proof, we use the uniform antimaximum principle for problem (4.3), see Theorem A.3 and Corollary A.4, both in the appendix.

Let us observe that

$$\frac{[u_\lambda + g_\lambda(\lambda, \cdot, u_\lambda)]\,\Phi_1}{\|u_\lambda + g_\lambda(\lambda, \cdot, u_\lambda)\|_{L^r(\partial\Omega)}} = \frac{[u_\lambda + g_\lambda(\lambda, \cdot, u_\lambda)]\,\Phi_1}{\|u_\lambda\|_{L^\infty(\partial\Omega)}}\ \frac{\|u_\lambda\|_{L^\infty(\partial\Omega)}}{\|u_\lambda + g_\lambda(\lambda, \cdot, u_\lambda)\|_{L^r(\partial\Omega)}}$$

but

$$\begin{aligned} &\frac{\|u_\lambda + g_\lambda(\lambda, \cdot, u_\lambda)\|_{L^r(\partial\Omega)}}{\|u_\lambda\|_{L^\infty(\partial\Omega)}} \le \\ &\quad \le \frac{\|u_\lambda\|_{L^r(\partial\Omega)}}{\|u_\lambda\|_{L^\infty(\partial\Omega)}} + \left\| \frac{g_\lambda(\lambda, \cdot, u_\lambda)}{u_\lambda}\ \frac{u_\lambda}{\|u_\lambda\|_{L^\infty(\partial\Omega)}} \right\|_{L^r(\partial\Omega)} \le C\left(1 + \|G_2\|_{L^r(\partial\Omega)}\right) := C_1 \end{aligned} \tag{4.4}$$

as $\lambda \to \sigma_1$. Hence

$$\liminf_{\lambda\to\sigma_1} \frac{[u_\lambda + g_\lambda(\lambda, \cdot, u_\lambda)]\,\Phi_1}{\|u_\lambda + g_\lambda(\lambda, \cdot, u_\lambda)\|_{L^r(\partial\Omega)}} \ge \liminf_{\lambda\to\sigma_1} \frac{[u_\lambda + g_\lambda(\lambda, \cdot, u_\lambda)]\,\Phi_1}{\|u_\lambda\|_{L^\infty(\partial\Omega)}}\ \frac{1}{C_1}$$

Moreover, taking into account that $\frac{u_\lambda}{\|u_\lambda\|_{L^\infty(\partial\Omega)}} \to \Phi_1$ in $L^\infty(\partial\Omega)$ as $\lambda \to \sigma_1$, we get

$$\begin{aligned} \int_{\partial\Omega} \liminf_{\lambda\to\sigma_1}\ \frac{[u_\lambda + g_\lambda(\lambda, \cdot, u_\lambda)]}{\|u_\lambda\|_{L^\infty(\partial\Omega)}}\ \Phi_1 &= \int_{\partial\Omega} \liminf_{\lambda\to\sigma_1}\ \left[1 + \frac{g_\lambda(\lambda, \cdot, u_\lambda)}{u_\lambda}\right]\ \frac{u_\lambda}{\|u_\lambda\|_{L^\infty(\partial\Omega)}}\Phi_1 \\ &\ge \int_{\partial\Omega} \liminf_{\lambda\to\sigma_1}\ \left[1 + \frac{g_\lambda(\lambda, \cdot, u_\lambda)}{u_\lambda}\right]\ |\Phi_1|^2 > 0 \end{aligned} \tag{4.5}$$

where we have used hypothesis ($H4$), see (4.1). Therefore

$$\int_{\partial\Omega} \liminf_{\lambda \to \sigma_1} \frac{[u_\lambda + g_\lambda(\lambda, \cdot, u_\lambda)]\,\Phi_1}{\|u_\lambda + g_\lambda(\lambda, \cdot, u_\lambda)\|_{L^r}} > 0 \tag{4.6}$$

and all the hypothesis of Corollary A.4 are fullfiled.

From hypothesis (3.10) in Theorem 3.4 and Lemma 3.3, we have $\mu_1 > \lambda$, for any λ close enough to σ_1, where $\mu_1 = \mu_1(-g_u(\lambda, \cdot, u_\lambda(\cdot)))$. Then , from Corollary A.4, there exists a constant C independent of λ such that $v > 0$ if $\mu_1 - C < \lambda < \mu_1$. This shows i).

ii) From Theorem 3.5 we have $\mu_1 < \lambda$ for any λ close enough to σ_1. Then, again from Corollary A.4, there exists a constant C independent of λ such that $v < 0$ if $\mu_1 + C > \lambda > \mu_1$. Hence, we get the result. $\square$

In a very similar way, we can prove the following result for the supercritical case.

Theorem 4.2 (Monotonicity for negative large equilibria)
i) If the conditions of Theorem 3.4 and (H4) hold, then $\dfrac{\partial u_\lambda}{\partial \lambda}(x) < 0$ *in* Ω *for negative solutions.*
ii) If the conditions of Theorem 3.5 and (H4) hold, then $\dfrac{\partial u_\lambda}{\partial \lambda}(x) > 0$ *in* Ω *for negative solutions.*

5 Bifurcation of the global attractors

In this section we want to analyze the behavior of the global dynamics of the flow defined by the evolutionary equation (1.1) as we cross the parameter value $\lambda = \sigma_1$, the first Steklov eigenvalue. As mentioned in Section 2, it is known that if $\lambda < \sigma_1$ then, the flow defined by (1.1) is dissipative and compact, hence it will have a global attractor, see [6, Section 7]. On the other hand, when $\lambda > \sigma_1$ the flow is not longer dissipative and we have initial conditions for which the solution of (1.1) grows without bounds (blows-up in infinite time). To see this we just need to take an initial condition $u_0(x) \equiv M$ a very large constant. Hence, the character of the global dynamics changes drastically when λ crosses this value of the parameter and we want to understand how this affects the behavior of the attractors.

We first start with the description of the global dynamics for $\lambda < \sigma_1$.

Lemma 5.1 *Assume that* $g(\lambda, x, u)$ *satisfies hypotheses (H1), (H2) and (H3). Also assume that* $\lambda < \sigma_1$*.*

Then the parabolic problem (1.1) has a global compact attractor $\mathcal{A}_\lambda \subset C(\bar\Omega)$*. Even more, there exist two extremal equilibria* $u_{m,\lambda} \le u_{M,\lambda}$ *in the sense that any other equilibria* φ *satisfies*

$$u_{m,\lambda}(x) \le \varphi(x) \le u_{M,\lambda}(x), \quad x \in \Omega.$$

Moreover, they bound the asymptotic dynamics of (1.1) in the sense that

$$u_{m,\lambda}(x) \le \liminf_{t \to \infty} |u(t, x, u_0)| \le \limsup_{t \to \infty} |u(t, x, u_0)| \le u_{M,\lambda}(x)$$

uniformly in $x \in \bar\Omega$ *and for* u_0 *in bounded sets of initial data. In particular for every* $\varphi \in \mathcal{A}_\lambda$ *we have* $u_{m,\lambda} \le \varphi \le u_{M,\lambda}$*.*

Proof. The existence of the attractor follows from Section 7 in [6]. Morever in that paper it was also proved that, for every $\varepsilon > 0$,

$$g(\lambda, x, u)u \le \varepsilon |h(x)| u^2 + D_\varepsilon |h(x)||u|$$

with some constant $D_\varepsilon > 0$ and h as in (2.1) and that denoting by φ_ε the unique solution of

$$\begin{cases} -\Delta\varphi + \varphi &= 0, & \text{in } \Omega \\ \frac{\partial \varphi}{\partial n} &= (\lambda + \varepsilon|h(x)|)\varphi + D_\varepsilon |h(x)|, & \text{on } \partial\Omega, \end{cases}$$

we have

$$\limsup_{t\to\infty} |u(t,x)| \le \varphi_\varepsilon(x), \quad \text{uniformly in} \quad x \in \Omega.$$

We now follow the arguments in [15], which roughly speaking state that considering φ_ε as an initial data in (1.1), the solution $u(t, \varphi_\varepsilon)$ decreases monotonically in time. Thus the maximal equilibrium $u_{M,\lambda}$ is the limit equilibrium of such solution. Arguing with $-\varphi_\varepsilon$ we get the minimal equiibria $u_{m,\lambda}$.□

Now we turn into the dynamics for $\lambda > \sigma_1$. In what follows we will concentrate in the case where supercritical bifurcation of both, positive an negative, equilibria occur. In particular, the bifurcating equilibria are unstable. To simplify the exposition we will assume that the function g is independent of the parameter λ, that is $g = g(x, u)$. Hence, we consider the problem

$$\begin{cases} u_t - \Delta u + u &= 0, & \text{in } \Omega, \quad t > 0 \\ \frac{\partial u}{\partial n} &= \lambda u + g(x,u), & \text{on } \partial\Omega,\ t > 0 \\ u(0,x) &= u_0(x), & \text{in } \Omega \end{cases} \tag{5.1}$$

As a matter of fact, we will assume in this section the following setting:

(S) i) The nonlinearity g satisfies hypotheses (H1), (H2) and (H3), see (2.1), (2.2), (2.3).

ii) For some $\alpha < 1$, there exist functions D_1, F_1 so that for $\lambda \to \sigma_1$, for sufficiently large $s > 0$ and $x \in \partial\Omega$ we have

$$\frac{|g_u(\cdot, s)|}{|s|^{\alpha-1}} \le D_1(x), \quad D_1 \in L^r(\partial\Omega) \text{ with } r > N-1 \tag{5.2}$$

$$\left| \frac{g(\sigma_1, \cdot, s) - s g_u(\sigma_1, \cdot, s)}{|s|^\alpha} \right| \le F_1(x), \qquad F_1 \in L^1(\partial\Omega). \tag{5.3}$$

iii) The following conditions hold

$$\overline{\mathbf{F}}_+ < 0 \qquad \text{and} \qquad \overline{\mathbf{F}}_- < 0. \tag{5.4}$$

Remark 5.2 *In particular, from Theorem 3.5 we have a supercritical bifurcation of both positive and negative branches of equilibria.*

Also, since g is independent of the parameter λ, hypothesis (H4) holds and we obtain the monotonicity in λ of the positive and negative branches of solutions of (1.2) bifurcating from infinity, see Theorem 4.2.

Hence, there exists a $\delta > 0$ small enough and $M > 0$, large enough, such that for all $\lambda \in (\sigma_1, \sigma_1 + \delta)$ there is a unique positive solution u_λ^+ with $\|u_\lambda^+\|_{L^\infty(\Omega)} > M$ and a unique negative solution u_λ^- with $\|u_\lambda^-\|_{L^\infty(\Omega)} > M$.

Let us denote by $T_\lambda(t)u_0$ the flow generated by (5.1), that is, $T_\lambda(t)u_0$ is the solution of (5.1) at time t. Observe that we make explicit the dependence of the flow on the parameter λ. An important result that will allow us to compare the solutions of (5.1) for two different values of λ is the following.

Lemma 5.3 *Let $u_0 > 0$. Then, if $\tau > 0$ is such that $T_\lambda(t)u_0 \geq 0$ for $t \in [0, \tau]$, then $T_\mu(t)u_0 \leq T_\lambda(t)u_0$ (resp. $T_\mu(t)u_0 \geq T_\lambda(t)u_0$) for each $\mu < \lambda$.*

Analogously, if $u_0 < 0$ and $T_\lambda(t)u_0 \leq 0$ for $t \in [0, \tau]$, then $T_\mu(t)u_0 \geq T_\lambda(t)u_0$ for each $\mu < \lambda$.

Proof. To show the result we just need to realize that if $T_\lambda(t)u_0 \geq 0$ then it is a supersolution for the corresponding problem with $\mu < \lambda$.□

We have the following,

Proposition 5.4 *With the setting given by (S), we have*
i) For $\lambda \in (\sigma_1 - \delta, \sigma_1)$ there exists a positive constant K such that

$$\sup_{\varphi \in \mathcal{A}_\lambda} \|\varphi\|_{C(\bar{\Omega})} \leq K, \quad \forall \lambda \in (\sigma_1 - \delta, \sigma_1)$$

where $\mathcal{A}_\lambda \subset C(\bar{\Omega})$ is the attractor for (5.1).
ii) For $\lambda \in (\sigma_1, \sigma_1 + \delta)$, if we define the open set

$$X_\lambda = \{\varphi \in C(\bar{\Omega}), u_\lambda^-(x) < \varphi(x) < u_\lambda^+(x)\},$$

then the flow given by (5.1) restricted to X_λ has also an attractor (a local attractor), that we denote $\mathcal{A}_\lambda \subset C(\bar{\Omega})$. Moreover, there exists a positive constant K such that

$$\sup_{\varphi \in \mathcal{A}_\lambda} \|\varphi\|_{C(\bar{\Omega})} \leq K, \quad \forall \lambda \in (\sigma_1, \sigma_1 + \delta).$$

iii) For the resonant case, $\lambda = \sigma_1$, the flow given by (5.1) also has an attractor $\mathcal{A}_{\sigma_1} \subset C(\bar{\Omega})$.

Remark 5.5 *Observe that for elliptic problems, Landesman-Lazer conditions establish the existence of equilibria for the resonant problem, see [12, 6]. In our case, part iii) of the previous proposition can be reinterpreted as follows: if the Landesman-Lazer conditions (5.4) hold, then the resonant problem has also an attractor.*

Proof. Let us start with the following important observation. Since all the equilibria bifurcating from infinity at σ_1 are contained in the bifurcating branches $u_\lambda^\pm$ which are supercritical, then, for $\delta > 0$ small there exists a constant $K > 0$ such that any other equilibria v_λ for the flow T_λ, for any $\sigma_1 - \delta < \lambda < \sigma_1 + \delta$, must satisfy $\|v_\lambda\|_{L^\infty(\Omega)} \leq K/2$.

From (5.4) the bifurcation of equilibria is supercritical. Hence, if $\sigma_1 - \delta < \lambda < \sigma_1$, the extremal solutions, $u_{m,\lambda}$ and $u_{M,\lambda}$, obtained in Lemma 5.1, satisfty the estimate $\|u_{m,\lambda}\|_{L^\infty(\Omega)}$, $\|u_{M,\lambda}\|_{L^\infty(\Omega)} \leq K$. The result follows from Lemma 5.1. This shows i).

For $\sigma_1 < \lambda < \sigma_1 + \delta$, it is clear that the *interval* $X_\lambda = \{\varphi \in C(\bar{\Omega}), u_\lambda^-(x) < \varphi(x) < u_\lambda^+(x)\}$ is invariant by T_λ and using the monotonicity in λ of the branches of equilibria $u_\lambda^\pm$, we have that for each $\sigma_1 < \mu < \lambda$, $X_\lambda \subset X_\mu$. If we denote by $X_{\sigma_1} = C(\bar{\Omega})$, that is, the whole space, we have that for any $\lambda_0 \in [\sigma_1, \sigma_1 + \delta)$,

$$X_{\lambda_0} = \bigcup_{\lambda_0 < \mu < \sigma_1 + \delta} X_\mu$$

Moreover, by Lemma 5.3, each of the sets X_μ, for $\lambda_0 < \mu < \sigma_1 + \delta$ is positively invariant by the flow $T_{\lambda_0}(t)$, that is $T_{\lambda_0}(t)X_\mu \subset X_\mu$, for $t > 0$. Using the monotonicity of the flow, we have $T_{\lambda_0}(t)X_\mu \subset T_{\lambda_0}(s)X_\mu$ for all $t > s$. In particular $T_{\lambda_0}(t)u_\mu^+ \leq u_\mu^+$, $(T_{\lambda_0}(t)u_\mu^- \geq u_\mu^-)$ for all $t > 0$ and $\lambda_0 < \mu < \sigma_1 + \delta$. Since the flow is gradient, see the comments at the end of Section 2, we will have that $T_{\lambda_0}(t)u_\mu^+$ and $T_{\lambda_0}(t)u_\mu^-$ will converge to the set of equilibria and in particular, there will exist a time $\tau > 0$, that will depend on μ, such that $T_{\lambda_0}(t)X_\mu \subset [-K, K] = \{\varphi \in C^0(\bar{\Omega}); |\varphi(x)| \leq K\}$ for all $\lambda_0 < \mu < \sigma_1 + \delta$. This shows the dissipativeness properties of the flow $T_{\lambda_0}(t)$, which in turn implies, with the compactness properties of the flow in Section 2, the existence of the attractor. $\square$

We can also provide some extra information in relation with the instability properties of the equilibria u_λ^+ and u_λ^- for $\sigma_1 < \lambda < \sigma_1 + \delta$.

Proposition 5.6 *For $\sigma_1 < \lambda < \sigma_1 + \delta$ we have*
i) The solutions u_λ^+ and u_λ^- have a unique unstable eigenvalue.
ii) For each initial data $\varphi \gneqq u_\lambda^+$, we have $\|T_\lambda(t)\varphi\|_{L^\infty(\Omega)} \to \infty$, as $t \to \infty$.
Analogously, for $\varphi \lneqq u_\lambda^-$, we have $\|T_\lambda(t)\varphi\|_{L^\infty(\Omega)} \to \infty$ as $t \to \infty$.
iii) The local attractors $\mathcal{A}_\lambda \subset C(\bar{\Omega})$ for the flow given by (5.1) restricted to X_λ, as in Proposition 5.4 have extremal equilibria $u_{m,\lambda} \leq u_{M,\lambda}$ in the sense that any other equilibria φ satisfies

$$u_{m,\lambda}(x) \leq \varphi(x) \leq u_{M,\lambda}(x), \quad x \in \Omega.$$

Moreover, they bound the asymptotic dynamics of (5.1) in the sense that

$$u_{m,\lambda}(x) \leq \liminf_{t\to\infty} |u(t, x, u_0)| \leq \limsup_{t\to\infty} |u(t, x, u_0)| \leq u_{M,\lambda}(x)$$

uniformly in $x \in \Omega$ and for u_0 in bounded sets of initial data B in X_λ such that

$$\inf_{\varphi\in B} \inf_{x\in\Omega} (u_\lambda^+(x) - \varphi(x)) > 0, \quad \inf_{\varphi\in B} \inf_{x\in\Omega} (\varphi(x) - u_\lambda^-(x)) > 0.$$

In particular for every $\varphi \in \mathcal{A}_\lambda$ we have $u_{m,\lambda} \leq \varphi \leq u_{M,\lambda}$.

Proof. i) The the linearization of the equation around u_λ^+ are given by (3.1), that is, if we define the potentials $b_\lambda(x) = -\lambda - g_u(\lambda, x, u_\lambda^+(x))$, we have

$$\left\{ \begin{array}{rcll} -\Delta\xi + \xi &=& \Lambda\xi, & \text{in } \Omega \\ \frac{\partial\xi}{\partial n} + b_\lambda(x)\xi &=& 0, & \text{on } \partial\Omega. \end{array} \right.$$

Then, with the notations in (2.5), we denote the eigenvalues as $\Lambda_i(-\lambda - g_u(\lambda, \cdot, u_\lambda^+(\cdot)))$, $i = 1, 2, \ldots$.

Observe that as $\lambda \to \sigma_1$, using (1.5) and (5.2) we have that $b_\lambda(\cdot) \to -\sigma_1$ in $L^r(\partial\Omega)$. Since $r > N - 1$, this convergence of the potentials guarantees the convergence of the eigenvalues, that is $\Lambda_i(b_\lambda(\cdot)) \to \Lambda_i(-\sigma_1)$, see Proposition A.2 in the Appendix. But we know that the first eigenvalue $\Lambda_1(-\sigma_1)$ is simple and since σ_1 is the first Steklov eigenvalue, we have $\Lambda_1(-\sigma_1) = 0$. In particular $\Lambda_2(-\sigma_1) > 0$. Hence, $\Lambda_2(b_\lambda(\cdot)) > 0$ for λ near σ_1.

ii) Observe that if $\varphi \gneqq u_\lambda^+$, then by the strong maximum principle, we have $T_\lambda(t)\varphi$ is strictly above u_λ^+ for some small time $t > 0$. Hence, without loss of generality we may assume that φ is strictly above u_λ^+. Hence, we have $\mu < \lambda$ close enough to λ with the property that $u_\lambda^+ < u_\mu^+ < \varphi$ and by comparison principles we get $T_\lambda(t)u_\mu^+ < T_\lambda(t)\varphi$. But, with Lemma 5.3, we have $u_\mu^+ = T_\mu(t)u_\mu^+ \leq T_\lambda(t)u_\mu^+$, which implies that $T_\lambda(t)u_\mu^+$ is nondecreasing in t. If $T_\lambda(t)u_\mu^+$ is bounded as $t \to +\infty$, by the gradient structure of the flow, it will have to converge to an equilibria but this is impossible since there is no equilibria above u_λ^+. Hence, $\|T_\lambda(t)u_\mu^+\|_{L^\infty(\Omega)} \to \infty$ and since $T_\lambda(t)u_\mu^+ \leq T_\lambda(t)\varphi$ we also have $\|T_\lambda(t)\varphi\|_{L^\infty(\Omega)} \to \infty$.

A similar argument is used to analyze the case $\varphi \lneqq u_\lambda^-$.

iii) The same argument as in ii) above shows that, given λ, if we chose $\mu > \lambda$ close enough we have $u_\lambda^+ > u_\mu^+ = T_\mu(t)u_\mu^+ \geq T_\lambda(t)u_\mu^+$, which implies that $T_\lambda(t)u_\mu^+$ is nonincreasing in t. Since the semigroup $T_\lambda(t)$ is a gradient system, the omega limit set of this nonincreasing, non stationary solution of (1.1), must be an equilibria in X_λ. This gives the maximal extremal equilibrium. Arguing similarly with u_μ^- we get the minimal extremal equilibrium.

If now $B \subset X_\lambda$ is as in the statement, then we can chose $\mu > \lambda$ such that $u_\mu^- \leq \varphi \leq u_\mu^+$ for all $\varphi \in B$ and the result follows.□

We are in a position now to prove the following result on the upper semicontinuity of the family of attractors $\mathcal{A}_\lambda$.

Proposition 5.7 *Assume the setting given by (S) and let $\lambda_0 \in (\sigma_1 - \delta, \sigma_1 + \delta)$.*

Then

$$dist(\mathcal{A}_\lambda, \mathcal{A}_{\lambda_0}) \longrightarrow 0, \quad as \quad \lambda \to \lambda_0, \tag{5.5}$$

where $dist(A, B) = \sup_{\varphi \in A} \inf_{\psi \in B} \|\varphi - \psi\|_{C(\bar{\Omega})}$.

Proof. Observe that for fixed $\lambda_0 \in (\sigma_1 - \delta, \sigma_1 + \delta)$ and for small $\varepsilon > 0$, we have that with the bounds obtained for $\mathcal{A}_\lambda$, we have that if $B_\varepsilon = \cup_{\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)} \mathcal{A}_\lambda$, then $B_\varepsilon \subset X_{\lambda_0}$ and B is attracted by $\mathcal{A}_{\lambda_0}$. Hence, for a fixed $\eta > 0$ small, we have the existence of $\tau = \tau(\eta) > 0$ such that

$$dist(T_{\lambda_0}(\tau)\varphi, \mathcal{A}_{\lambda_0}) \leq \frac{\eta}{2}, \quad \forall \varphi \in B$$

Choosing ε smaller if necessary, we have that from (5.6) below $\|T_\lambda(\tau, \varphi) - T_{\lambda_0}(\tau, \varphi)\|_{C(\bar{\Omega})} \leq \frac{\eta}{2}$ for all $\varphi \in B_\varepsilon$ and therefore

$$dist(T_\lambda(\tau)\varphi, \mathcal{A}_{\lambda_0}) \leq \eta, \quad \forall \varphi \in \mathcal{A}_\lambda, \forall \lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$$

Using now the invariance of the attractor $\mathcal{A}_\lambda$ under the flow T_λ, that is $T_\lambda(\tau)\mathcal{A}_\lambda = \mathcal{A}_\lambda$, we get that

$$dist(\mathcal{A}_\lambda, \mathcal{A}_{\lambda_0}) \leq \eta, \quad \forall \lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$$

which shows (5.5).□

To conclude we prove the result used above.

Proposition 5.8 *Assume g in (5.1) satisfies (H1) and (H2). Then, for each λ_0, $0 < t_0 < t_1$ and each bounded set $B \subset C(\bar{\Omega})$, we have*

$$\sup_{t\in[t_0,t_1]} \sup_{\varphi\in B} \|T_\lambda(t)\varphi - T_{\lambda_0}(t)\varphi\|_{C(\bar{\Omega})} \to 0, \quad \text{as } \lambda \to \lambda_0 \tag{5.6}$$

Proof. Let us denote by $M = \sup_{\varphi\in B} \|\varphi\|_{L^\infty(\Omega)}$, so that $-M \leq \varphi(x) \leq M$ for each $\varphi \in B$. Moreover, since from $(H1)$ and $(H2)$ the nonlinearity g is sublinear, we have that

$$g(x,u)u \leq h(x)u^2 + Dh(x)|u|,$$

with $D > 0$ and $h(x)$ as in (2.1), see [6]. This implies that if we take $\delta > 0$ and define U the solution of the linear problem

$$\begin{cases} U_t - \Delta U + U &= 0, & \text{in } \Omega, \quad t > 0 \\ \frac{\partial U}{\partial n} &= (\lambda_0 + \delta + h(x))U + Dh(x) & \text{on } \partial\Omega,\ t > 0 \\ U(0,x) &= M, & \text{in } \Omega \end{cases}$$

then $U(t,x) > 0$ and by comparison results $|T_\lambda(t)\varphi(x)| \leq U(t,x)$, $t > 0$, $x \in \bar{\Omega}$, $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$. In particular, since $h \in L^r(\partial\Omega)$ with $r > N-1$, the solution of (5.1) is bounded uniformly for all $t \in [t_0, t_1]$, $\varphi \in B$ and $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$, that is, we have

$$\|T_\lambda(t)\varphi\|_{L^\infty(\Omega)} \leq C, \quad t > 0,\ \varphi \in B,\ \lambda \in (\lambda_0 - \delta, \lambda_0 + \delta). \tag{5.7}$$

Moreover, with the regularization properties of the parabolic equation we will have that there exists a constant C such that

$$\|T_\lambda(t)\varphi\|_{C^\alpha(\bar{\Omega})} \leq C, \quad t \in [t_0, t_1],\ \varphi \in B,\ \lambda \in (\lambda_0 - \delta, \lambda_0 + \delta). \tag{5.8}$$

If we denote by $T_\lambda(t)\varphi = u_\lambda(t)$, $T_{\lambda_0}(t)\varphi = u_{\lambda_0}(t)$ and $w = u_\lambda - u_{\lambda_0}$, we have that w satisfies

$$\begin{cases} w_t - \Delta w + w &= 0, & \text{in } \Omega, \quad t > 0 \\ \frac{\partial w}{\partial n} &= \lambda_0 w + (\lambda - \lambda_0)u + g(x,u_\lambda) - g(x,u_{\lambda_0}) & \text{on } \partial\Omega,\ t > 0 \\ w(0,x) &= 0, & \text{in } \Omega. \end{cases}$$

Multiplying by w, integrating by parts and operating in the resulting identity, we get

$$\frac{d}{dt}\int_\Omega w^2 + \int_\Omega |\nabla w|^2 + \int_\Omega |w|^2 = \lambda_0 \int_{\partial\Omega} w^2 + (\lambda - \lambda_0)\int_{\partial\Omega} u_\lambda w + \int_{\partial\Omega} (g(x,u_\lambda) - g(x,u_{\lambda_0}))w$$

Applying estimate (5.7) to u_λ together with Young inequality to the second term on the right hand side and the fact that g is Lipschitz on bounded sets of u, we have

$$\frac{d}{dt}\int_\Omega w^2 + \int_\Omega |\nabla w|^2 + \int_\Omega |w|^2 \leq |\lambda - \lambda_0| + C\int_{\partial\Omega} w^2$$

for some constant C which is independent of λ, and $\varphi \in B$. Using the standard Sobolev trace inequality

$$\int_{\partial\Omega} w^2 \leq \varepsilon \int_\Omega |\nabla w|^2 + C_\varepsilon \int_\Omega |w|^2$$

we get

$$\frac{d}{dt}\int_\Omega w^2 \le |\lambda-\lambda_0| + C\int_\Omega w^2$$

and elementary integration shows, using $w(0)=0$,

$$\int_\Omega w^2 \le |\lambda-\lambda_0|\frac{e^{Ct}-1}{C}$$

which shows the L^2 convergence, that is

$$\sup_{t\in[t_0,t_1]}\sup_{\varphi\in B}\|T_\lambda(t)\varphi - T_{\lambda_0}(t)\varphi\|_{L^2(\Omega)} \to 0, \quad \text{as } \lambda\to\lambda_0$$

This L^2-convergence together with the uniform Hölder estimate given by (5.8) and an elementary compactness argument, shows the convergence in the uniform topology, which shows (5.6). □

A A uniform Antimaximum Principle

Let us consider a family of nonhomogeneous linear Steklov problems containing a potential at the boundary of the form $b_0(x)+\eta(x)$ where $b_0(\cdot)\in L^r(\partial\Omega)$ is a fixed potential and $\eta(\cdot)\in L^r(\partial\Omega)$ will be small in $L^r(\partial\Omega)$ with $r>N-1$, that is

$$\left\{\begin{array}{rcll} -\Delta u + u &=& 0, & \text{in } \Omega\\ \frac{\partial u}{\partial n} + [b_0(x)+\eta(x))]u &=& \lambda u + g(x), & \text{on } \partial\Omega \end{array}\right. \tag{A.1}$$

We will denote by $\mu_i^\eta := \mu_i(b_0+\eta)$ and $\varphi_i^\eta := \varphi_i(b_0+\eta)$, $i=1,2,\ldots$, (see the notation of (2.5)), that is, the Steklov eigenvalues and eigenfunctions of the problem

$$\left\{\begin{array}{rcll} -\Delta\varphi + \varphi &=& 0, & \text{in } \Omega\\ \frac{\partial\varphi}{\partial n} + [b_0(x)+\eta(x)]\varphi &=& \mu\varphi, & \text{on } \partial\Omega \end{array}\right. \tag{A.2}$$

so that μ_i^0 and φ_i^0, $i=1,2,\ldots$, are the Steklov eigenvalue and eigenfunction of the problem

$$\left\{\begin{array}{rcll} -\Delta\varphi_i^0 + \varphi_i^0 &=& 0, & \text{in } \Omega\\ \frac{\partial\varphi_i^0}{\partial n} + b_0(x)\varphi_i^0 &=& \mu_i^0\varphi_i^0, & \text{on } \partial\Omega. \end{array}\right. \tag{A.3}$$

We start proving a result on the behavior of the solutions of (A.1) and of the spectra of (A.2). This result is used in several instances in this paper and it will also be needed to prove the uniform antimaximum principle.

Proposition A.1 *Let us consider a family of potentials $\eta_n\in L^r(\partial\Omega)$ for some $r>N-1$, satisfying $\eta_n \rightharpoonup 0$, weakly in $L^r(\partial\Omega)$. Denote by $S_\eta : L^r(\partial\Omega)\to L^r(\partial\Omega)$, the solution operator of (A.1) with $\lambda=0$, that is $S_\eta(g)=\gamma(u)$, where u is the solution of (A.1) with $\lambda=0$ and $\gamma(\cdot)$ is the trace operator. Then, there exists a large enough constant $a>0$ such that S_a and $S_{a+\eta_n}$ are well defined and*

$$\|S_{a+\eta_n} - S_a\|_{\mathcal{L}(L^r(\partial\Omega),L^r(\partial\Omega))} \to 0, \ \text{as } n\to+\infty. \tag{A.4}$$

Moreover, we have the convergence of eigenvalues and eigenfunctions, that is $\mu_i^{\eta_n} \to \mu_i^0$ as $n \to +\infty$ for all $i = 1, 2, ..,$ and in particular

$$\varphi_1^{\eta_n} \to \varphi_1^0, \text{ in } H^1(\Omega),\, C^\alpha(\bar\Omega) \tag{A.5}$$

for some $\alpha > 0$.

Proof. From the weak convergence of η_n we obtain that the sequence is bounded in $L^r(\partial\Omega)$ and, therefore, the sequence of potentials $b_0 + \eta_n$ is also bounded in $L^r(\partial\Omega)$.

The solution of (A.1) with $\lambda = 0$ is obtained applying Lax-Milgram to the bilinear form defined in $H^1(\Omega)$:

$$a_\eta(u, v) = \int_\Omega \nabla u \nabla v + uv + \int_{\partial\Omega} (b_0 + \eta) uv$$

Using the boundedness of the potentials $b_0 + \eta_n$ in $L^r(\partial\Omega)$, uniformly in n, it is not difficult to see that we can choose $a > 0$ large enough such that the bilinear forms a_{η_n+a} are uniformly coercive in $H^1(\Omega)$. This implies that we can solve in a unique way problem (A.1) with $\lambda = -a$. Elliptic regularity theory guarantees that this solution lies in better spaces than $H^1(\Omega)$ (see [6]), in particular in $H^1(\Omega) \cap C^\beta(\bar\Omega)$ for certain $\beta > 0$. Moreover, since $r > N - 1$, the following estimate can be obtained,

$$\|u_n\|_{H^1(\Omega)} + \|u_n\|_{C^\beta(\bar\Omega)} \le C\|g\|_{L^r(\bar\Omega)} \tag{A.6}$$

with C independent of n, see Lemma 2.1 in [6]. In particular, the operators S_{η_n+a} and S_a are well defined.

In order to show (A.4) we will prove that if g_n is a bounded sequence of functions in $L^r(\partial\Omega)$ such that $g_n \rightharpoonup g$ weakly in $L^r(\Omega)$ then $S_{\eta_n+a}(g_n) \to S_a(g)$ in $L^r(\partial\Omega)$. If this were not true, we could take a subsequence, that we denote again by n, such that $\|S_{\eta_n+a}(g_n) - S_a(g)\|_{L^r(\partial\Omega)} \ge \rho > 0$ for some fix ρ. But from (A.6) we obtain that there exists another subsequence, denoted still by n, and a function $u \in H^1(\Omega) \cap C^\beta(\bar\Omega)$ such that $u_n \to u$ weakly in $H^1(\Omega)$ and strongly in $C^\alpha(\bar\Omega)$ with $0 < \alpha < \beta$. This convergence will permit us to pass to the limit in the weak formulations of the problems and obtain that u is actually the solution of the limit problem, that is (A.1) with $\lambda = -a$. Hence, $S_{\eta_n+a}(g_n) = \gamma(u_n) \to \gamma(u) = S_a(g)$ in $C^\alpha(\partial\Omega)$ and in particular in $L^r(\partial\Omega)$. This shows (A.4).

The convergence of the eigenvalues and eigenfunctions is a direct consequence of (A.4). Moreover, (A.5) is obtained from the simplicity of the first eigenvalue and elliptic regularity theory. We refer to [11] for a general reference. Also, see [3] for an example in other context on how to obtain the behavior of the spectra from the convergence of the resolvent operators.$\square$

In a very similar way we have,

Proposition A.2 *Let us consider a family of potentials $\eta_n \in L^r(\partial\Omega)$ for some $r > N - 1$, satisfying $\eta_n \rightharpoonup 0$, weakly in $L^r(\partial\Omega)$. Denote by $T_{\eta,c} : L^2(\Omega) \to L^2(\Omega)$, the solution operator of*

$$\begin{cases} -\Delta u + cu = f, & \text{in } \Omega \\ \frac{\partial u}{\partial n} + [b_0(x) + \eta(x))]u = 0, & \text{on } \partial\Omega, \end{cases} \tag{A.7}$$

that is $T_{\eta,c}(f) = u$, where u is the solution of (A.7). Then, there exists a large enough constant $c > 0$ such that $T_{\eta_n,c}$ and $T_{0,c}$ are well defined and

$$\|T_{\eta_n,c} - T_{0,c}\|_{\mathcal{L}(L^2(\Omega),L^2(\Omega))} \to 0, \text{ as } n \to +\infty. \tag{A.8}$$

Moreover, we have the convergence of eigenvalues and eigenfunctions, that is, with the notations or (2.5), $\Lambda_i(b+\eta_n) \to \Lambda_i(b)$ as $n \to +\infty$ for all $i = 1, 2, ..,$ and similarly for the eigenfunctions.

Proof. The proof follows the same ideas as the proof of Proposition A.1. To show (A.8) we pass to the limit in the weak formulation of (A.7) and use elliptic regularity theory to show that the convergence is in stronger norms. The convergence of the eigenvalues and eigenfunctions follows from (A.8), see again [11]. □

Now, we want to analyze the behavior of the solutions of (A.1) with λ varying in a neighborhood of μ_1^0 and assuming that $\|\eta\|_{L^r(\partial\Omega)}$ is small. As a matter of fact, we can show:

Theorem A.3 *There exist three constants $\eta_0, d_0, M > 0$ such that for every function $\eta \in L^r(\partial\Omega)$ with $\|\eta\|_{L^r(\partial\Omega)} \leq \eta_0$ and every function $g \in L^r(\partial\Omega)$ with $r > N-1$ and $\displaystyle\int_{\partial\Omega} g\,\varphi_1^\eta > 0$ we have*

i) if $\lambda \in \left(\mu_1^\eta, \mu_1^\eta + M\,\dfrac{\int_{\partial\Omega} g\,\varphi_1^\eta}{\|g\|_{L^r(\partial\Omega)}}\right) \cap I$ *then* $u < 0$,

ii) if $\lambda \in \left(\mu_1^\eta - M\,\dfrac{\int_{\partial\Omega} g\,\varphi_1^\eta}{\|g\|_{L^r(\partial\Omega)}}, \mu_1^\eta\right) \cap I$, *then* $u > 0$.

where $I = [\mu_1^0 - d_0, \mu_1^0 + d_0]$ and u is the solution of (A.1) .

Proof. For each $\eta \in L^r(\partial\Omega)$ fixed, we consider

$$L^r(\partial\Omega) = \operatorname{span}[\varphi_1^\eta] \oplus \operatorname{span}[\varphi_1^\eta]^\perp, \tag{A.9}$$

where

$$\operatorname{span}[\varphi_1^\eta]^\perp := \left\{u \in L^r(\partial\Omega) : \int_{\partial\Omega} u\,\varphi_1^\eta = 0\right\} \tag{A.10}$$

and therefore, for every $g \in L^r(\partial\Omega)$ with $r > N-1$ there exists a unique decomposition

$$g = a_0(\eta)\varphi_1^\eta + g_1^\eta, \quad \text{where} \quad a_0(\eta) := \frac{\int_{\partial\Omega} g\,\varphi_1^\eta}{\int_{\partial\Omega} |\varphi_1^\eta|^2}, \quad \text{and} \quad \int_{\partial\Omega} g_1^\eta \varphi_1^\eta = 0. \tag{A.11}$$

The well known Fredholm Alternative states that the linear problem (A.1) for $\lambda \in \mathbb{R}$ does not have solution if $\lambda \in \{\mu_i^\eta\}_{i=1}^\infty$ and has a unique solution if $\lambda \neq \mu_i^\eta$, for all $i = 1, 2 \ldots$. The solution u in the latter case has a unique decomposition

$$u = \frac{a_0(\eta)}{\mu_1^\eta - \lambda}\,\varphi_1^\eta + u_1, \quad \text{with} \quad \int_{\partial\Omega} u_1\varphi_1^\eta = 0, \tag{A.12}$$

where $a_0(\eta)$ is defined in (A.11) and $u_1 = u_1(\eta, \lambda)$ solves the following problem

$$\left\{\begin{array}{rcll} -\Delta u_1 + u_1 &=& 0, & \text{in } \Omega \\ \frac{\partial u_1}{\partial n} + [b_0(x) + \eta(x)]\,u_1 &=& \lambda u_1 + g_1^\eta, & \text{on } \partial\Omega. \end{array}\right. \tag{A.13}$$

Moreover, by the decomposition of g, see (A.11), $u_1 \in \operatorname{span}[\varphi_1^\eta]^\perp$. By hypothesis and from the Fredholm Alternative, it is already known that the linear problem (A.13) has a unique solution u_1 in $\operatorname{span}[\varphi_1^\eta]^\perp$.

From the continuous dependence of the Steklov eigenvalues with respect to the potential given by Proposition A.1, we know that we have that $\mu_i^\eta \to \mu_i^0$ for all $i = 1, 2. \dots$ and

$$\varphi_1^\eta \to \varphi_1^0 \quad \text{in} \quad C^\alpha(\overline{\Omega}\,) \quad \text{ for some } \quad 0 < \alpha < 1, \quad \text{ as } \|\eta\|_{L^r(\partial\Omega)} \to 0. \tag{A.14}$$

which implies that we can choose $\tilde{\eta}_0 > 0$ small such that

$$\min_{x\in\bar{\Omega}} \frac{\varphi_1^\eta(x)}{\int_{\partial\Omega}|\varphi_1^\eta|^2} \geq \frac{1}{2} \min_{x\in\bar{\Omega}} \frac{\varphi_1^0(x)}{\int_{\partial\Omega}|\varphi_1^0|^2} > 0, \text{ for } \|\eta\|_{L^r(\partial\Omega)} \leq \tilde{\eta}_0. \tag{A.15}$$

Let $d_0 = (\mu_2^0 - \mu_1^0)/2 > 0$ and let us consider now $0 < \eta_0 \leq \tilde{\eta}_0$ small enough with the property that for each $\eta \in L^r(\partial\Omega)$ with $\|\eta\|_{L^r(\partial\Omega)} \leq \eta_0$, we have $[\mu_1^0 - d_0, \mu_1^0 + d_0] \cap \{\eta\}_{i=1}^\infty = \mu_1^\eta$.

Let us define the set $E = \{(\lambda, \eta) \in [\mu_1^0 - d_0, \mu_1^0 + d_0] \times L^r(\partial\Omega)$ with $\|\eta\|_{L^r(\partial\Omega)} \leq \eta_0$ and $\lambda \neq \mu_1^\eta\}$

We will next prove that for a fixed $g \in L^r(\partial\Omega)$, $u_1 = u_1(\lambda, \eta)$ is uniformly bounded for any $(\lambda, \eta) \in E$.

Let us argue by contradiction. If this is not the case, then there exists a sequence $(\lambda_n, \eta_n) \in E$ such that $\|u_1(\lambda_n, \eta_n)\|_{L^\infty(\partial\Omega)} \to \infty$. Taking another subsequence if necessary, we may assume that there exists $\eta \in L^r(\partial\Omega)$ such that $\eta_n \rightharpoonup \eta$, weakly in $L^r(\partial\Omega)$. Applying Proposition A.1 we get that $\mu_1^{\eta_n} \to \mu_1^\eta$ and $\varphi_1^{\eta_n} \to \varphi_1^\eta$ in $C^\alpha(\bar{\Omega})$. Arguing as in [6, Proposition 3.1], we get that necessarily this sequence must satisfy $\lambda_n \to \mu_1^\eta$ and, at least for another subsequence, that we denote the same, $\|\frac{u_1(\lambda_n,\eta_n)}{\|u_1(\lambda_n,\eta_n)\|_{L^\infty(\partial\Omega)}} - \varphi_1^\eta\|_{L^\infty(\Omega)} \to 0$. This is in contradiction with the fact that $u_1(\lambda) \in \operatorname{span}[\varphi_1^{\eta_n}]^\perp$ and the convergence (A.14).

Let us now define a family of operators $T(\lambda, \eta) : L^r(\partial\Omega) \to L^\infty(\Omega)$ for $(\lambda, \eta) \in E$, by $T(\lambda, \eta)(g) := u_1(\lambda, \eta)$. From elliptic regularity, $T(\lambda, \eta)$ is continuous. Moreover $\|T(\lambda, \eta)(g)\|_{L^\infty(\Omega)} \leq C(g)$ for all $(\lambda, \eta) \in E$. Therefore, applying the uniform boundedness principle, there exists a constant C_1 such that

$$\|u_1(\lambda, \eta)\|_{L^\infty(\partial\Omega)} \leq C_1\|g\|_{L^r(\partial\Omega)} \quad \text{for any } (\lambda, \eta) \in E. \tag{A.16}$$

Consider the case $\mu_1^\eta < \lambda$. From (A.12) and (A.16) we have that for $(\lambda, \eta) \in E$ we have $u \leq \frac{a_0(\eta)}{\mu_1^\eta - \lambda}\, \varphi_1^\eta + C_1\|g\|_{L^r}$. From here, if we define $C(\eta) := \min_{x\in\overline{\Omega}} \varphi_1^\eta(x) / \big(C_1 \int_{\partial\Omega}|\varphi_1^\eta|^2\big)$, we obtain that for $(\lambda, \eta) \in E$, if $0 < \lambda - \mu_1^\lambda < C(\eta)\, \dfrac{\int_{\partial\Omega} g\, \varphi_1^\eta}{\|g\|_{L^r}}$, then $u < 0$.

Now, taking into account (A.15) we have

$$C(\eta) \geq \frac{1}{2C_1} \min_{x\in\bar{\Omega}} \frac{\varphi_i^0(x)}{\int_{\partial\Omega}|\varphi_1^0|^2} := M > 0, \text{ for } \|\eta\|_{L^r(\partial\Omega)} \leq \eta_0$$

from where i) follows. The other inequality is obtained in a similar way. □

Let us finally consider a family of nonhomogeneous linear Steklov problems with a variable nonhomogeneous term at the boundary g depending on the parameter λ

$$\left\{ \begin{array}{rcll} -\Delta u + u &=& 0, & \text{in } \Omega \\ \frac{\partial u}{\partial n} + [b_0(x) + \eta(\lambda, x)]u &=& \lambda u + g(\lambda, x), & \text{on } \partial\Omega \end{array} \right. \tag{A.17}$$

where $g(\lambda, \cdot) \in L^r(\partial\Omega)$ and $b_0 + \eta(\lambda, \cdot) \in L^r(\partial\Omega)$. We will also assume that $\|\eta(\lambda, \cdot)\|_{L^r(\partial\Omega)} \to 0$ as $\lambda \to \mu_1^0$,

Corollary A.4 *Assume that the following hypothesis holds*

$$\|\eta(\lambda,\cdot)\|_{L^r(\partial\Omega)} \to 0, \qquad as \quad \lambda \to \mu_1^0. \tag{A.18}$$

Assume also that $\|g(\lambda,\cdot)\|_{L^r(\partial\Omega)} \neq 0$ *for all* $\lambda \in [\mu_1^0 - \delta_0, \mu_1^0 + \delta_0]$ *for some* $\delta_0 > 0$ *and that*

$$\liminf_{\lambda \to d_0} \int_{\partial\Omega} \frac{g(\lambda,\cdot)\,\varphi_1^0}{\|g(\lambda,\cdot)\|_{L^r(\partial\Omega)}} > 0. \tag{A.19}$$

Then there exist constants $\delta, \tilde{M} > 0$ *such that*

i) if $\lambda \in \left(\mu_1^{\eta(\lambda)}, \mu_1^{\eta(\lambda)} + \tilde{M}\right) \cap I$ *then* $u < 0$,

ii) if $\lambda \in \left(\mu_1^{\eta(\lambda)} - \tilde{M}, \mu_1^{\eta(\lambda)}\right) \cap I$, *then* $u > 0$.

where $I = [\mu_1^0 - \delta, \mu_1^0 + \delta]$ *and* u *is the solution of (A.17) .*

Proof. Define $\tilde{g}(\lambda,\cdot) = g(\lambda,\cdot)/\|g(\lambda,\cdot)\|_{L^r(\partial\Omega)}$ and $\tilde{u} = u/\|g(\lambda,\cdot)\|_{L^r(\partial\Omega)}$ so that $\tilde{u}$ satisfies

$$\left\{\begin{array}{rcll} -\Delta\tilde{u} + \tilde{u} &=& 0, & \text{in } \Omega \\ \frac{\partial\tilde{u}}{\partial n} + [b_0(x) + \eta(\lambda,x)]\tilde{u} &=& \lambda\tilde{u} + \tilde{g}(\lambda,x), & \text{on } \partial\Omega \end{array}\right. \tag{A.20}$$

From the convergence of $\varphi_1^{\eta(\lambda)}$ to φ_1^0 stated in (A.14) and from (A.19) we get

$$\liminf_{\lambda\to\mu_1^0} \int_{\partial\Omega} \tilde{g}(\lambda,\cdot)\,\varphi_1^{\eta(\lambda)} \geq \liminf_{\lambda\to\mu_1^0} \int_{\partial\Omega} \tilde{g}(\lambda,\cdot)\,[\varphi_1^{\eta(\lambda)} - \varphi_1^0] + \liminf_{\lambda\to\mu_1^0} \int_{\partial\Omega} \tilde{g}(\lambda,\cdot)\,\varphi_1^0 > 0$$

from where we obtain that there exists $a_0 > 0$ and $\delta > 0$ such that for $\lambda \in [\mu_1^0 - \delta, \mu_1^0 + \delta]$ we have

$$\int_{\partial\Omega} \tilde{g}(\lambda,\cdot)\,\varphi_1(\lambda,\cdot) \geq a_0, \quad \lambda \in [\mu_1^0 - \delta, \mu_1^0 + \delta].$$

Now the result is a consequence of the theorem above. $\square$

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